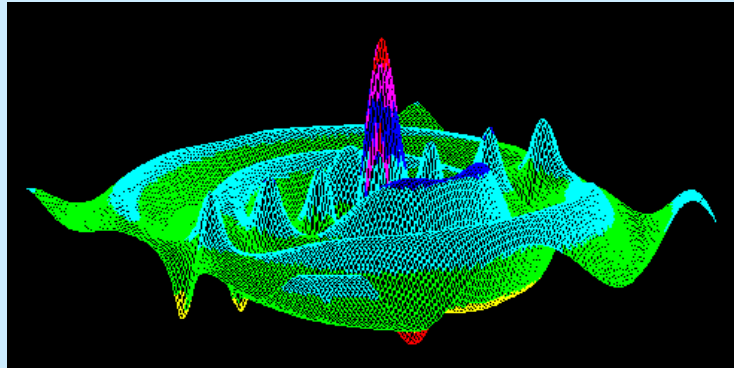


Detecting Objects in a layered medium



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1. The Mathematical model

Let $R_+^2 = \{\mathbf{x} \in R^2 | x_2 \geq 0\}$, where $\mathbf{x} = (x_1, x_2)$. $n(\mathbf{x})$ is defined on R_+^2 with the following properties:

Let

$$n_0(x_2) = \begin{cases} n_0, & \text{for } 0 < x_2 < h, \\ 1, & \text{for } h < x_2 < \infty, \end{cases} \quad (1.1)$$

where $n_0 > 1$ is a constant. We assume that the inhomogeneity is contained in a bounded domain $\Omega \in R_h^2 = \{\mathbf{x} \in R^2 | 0 < x_2 < h\}$ with C^2 boundary having outward normal vector ν .

$$n(\mathbf{x}) = n_0(x_2), \text{ for } \mathbf{x} \notin \Omega. \quad (1.2)$$

We consider the following problem: given point source or plane wave incident wave u^i satisfying

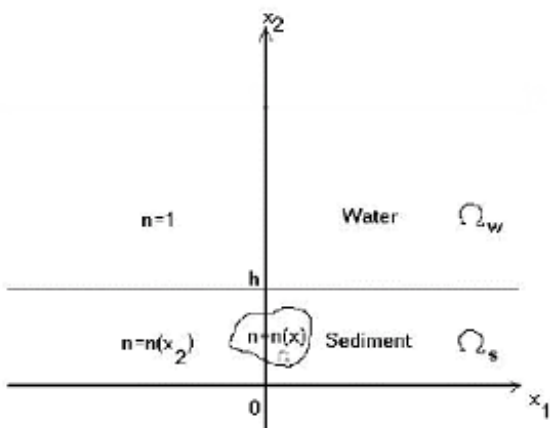
$$\Delta u^i + k^2 n_0^2(x_2) u^i = f(\mathbf{x}), \text{ in } R_+^2, \quad (1.3)$$

find the total field $u = u^s + u^i$ such that

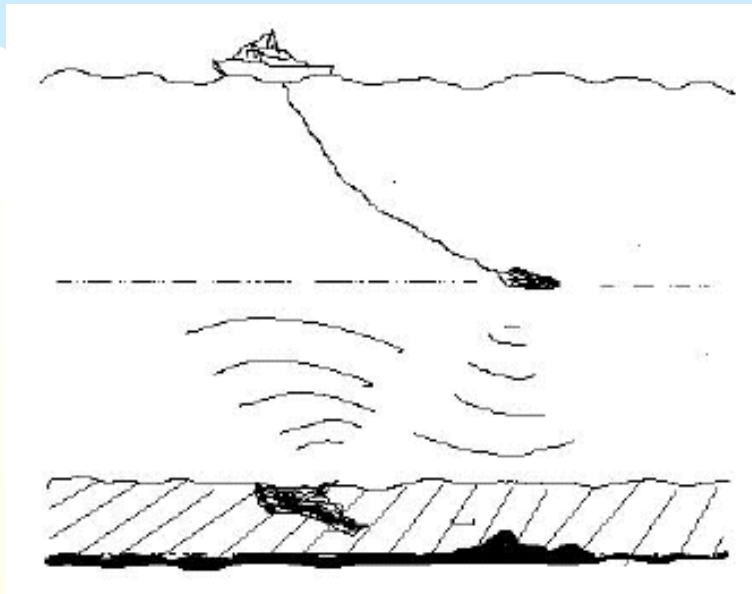
$$\Delta u^s + k^2 n^2(\mathbf{x}) u^s = 0, \text{ , in } R_+^2, \quad (1.4)$$

$$u(\mathbf{x}) = u^s(\mathbf{x}) + u^i(\mathbf{x}) = 0, \text{ when } x_2 = 0, \quad (1.5)$$

$$\rho_0 u^+(\mathbf{x}) = \rho u^-(\mathbf{x}), \quad \frac{\partial u^+(\mathbf{x})}{\partial \nu} = \frac{\partial u^-(\mathbf{x})}{\partial \nu}, \text{ on } \partial\Omega. \quad (1.6)$$



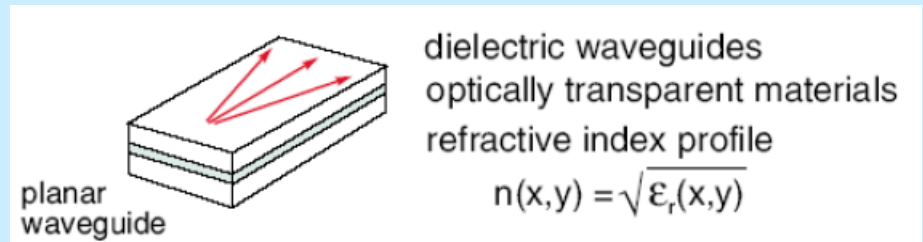
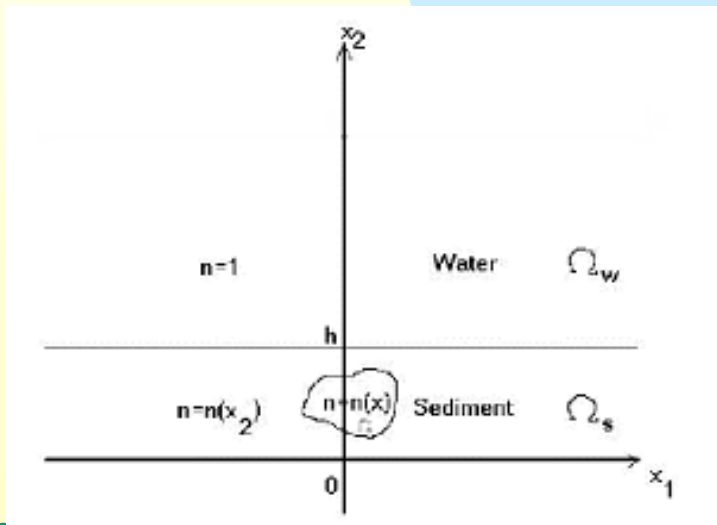
Sound wave



Electric magnetic wave



Light wave



2.1: Green's function and its far-field pattern

A function $G(\cdot; x_1^0, x_2^0)$ is the outgoing Green's function from the source at $\mathbf{x}^0 = (x_1^0, x_2^0)$ for the time-harmonic wave in a layered half-space, if $G(x_1, x_2; x_1^0, x_2^0)$ satisfies

$$\Delta G + k^2 n_0^2(x_2)G = -\delta(|\mathbf{x} - \mathbf{x}^0|), \text{ in } R_+^2 \quad (2.1)$$

in the generalized function sense;

$$G(x_1, 0; x_1^0, x_2^0) = 0, \quad (2.2)$$

$$G(x_1, x_2; x_1^0, x_2^0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{G}(\xi, x_2; x_1^0, x_2^0) e^{i\xi|x_1 - x_1^0|} d\xi, \quad (2.3)$$

where $\tilde{G}(\xi, x_2; x_1^0, x_2^0)$ satisfies

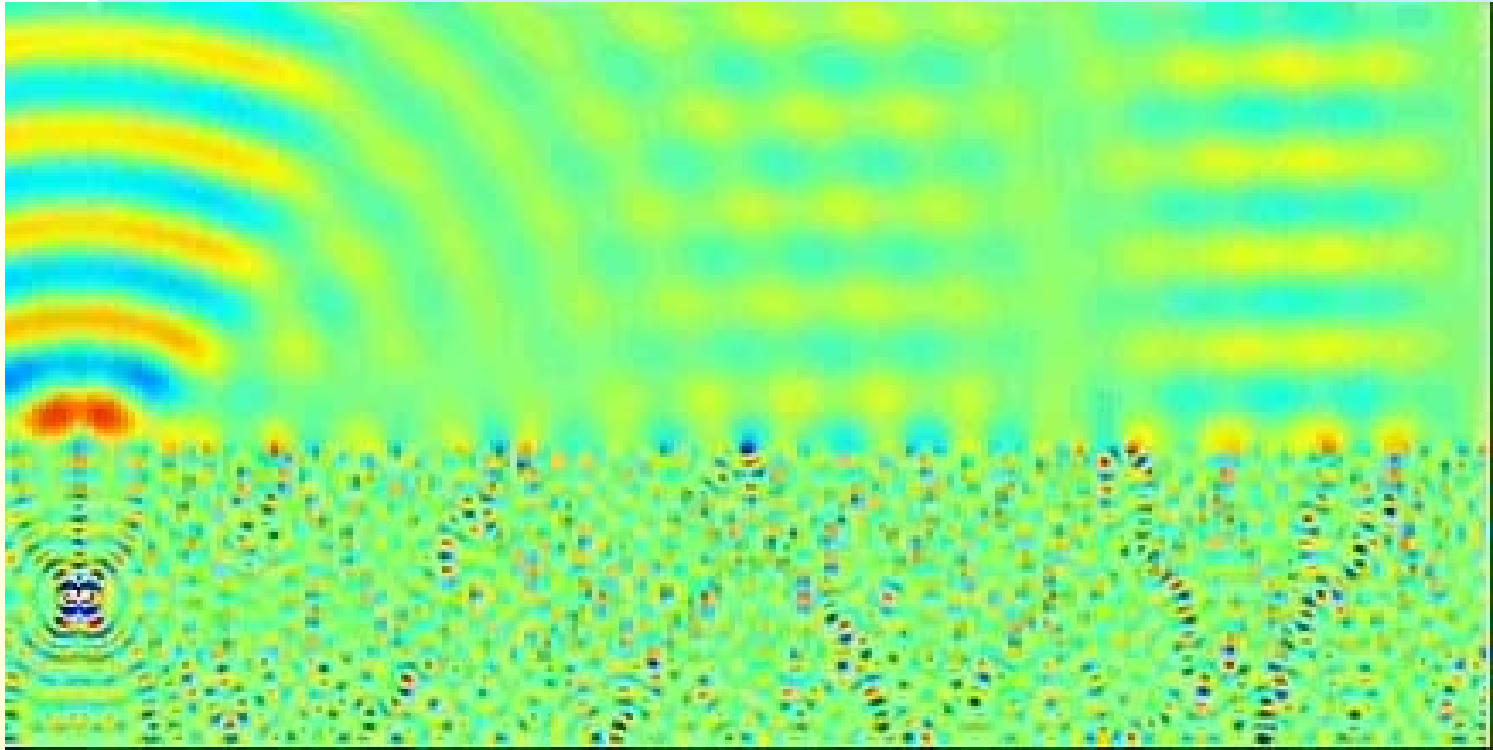
$$\tilde{G}''(x_2) + [k^2 n_0^2(x_2) - \xi^2] \tilde{G}(x_2) = -\delta(x_2 - x_2^0), \quad 0 < x_2 < \infty, \quad (2.4)$$

$$\tilde{G}(\xi, 0; x_1^0, x_2^0) = 0, \quad (2.5)$$

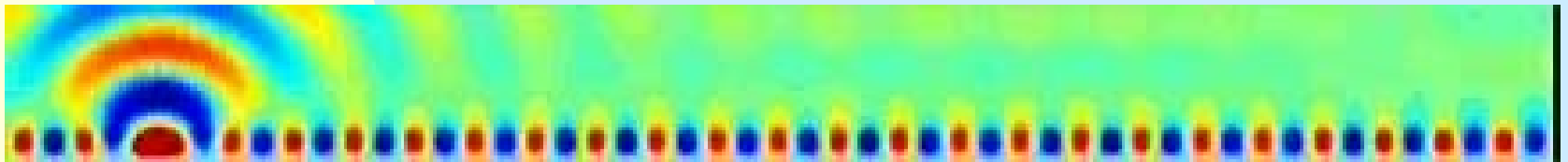
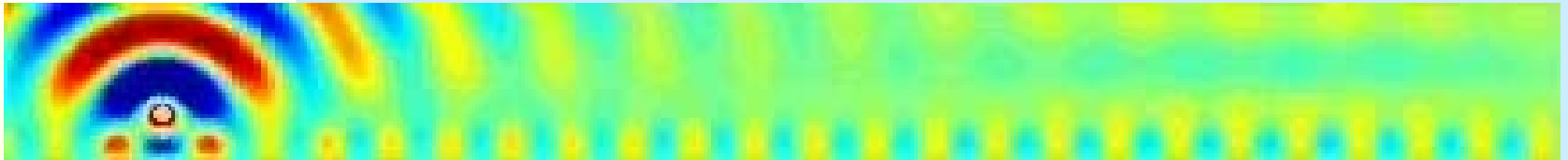
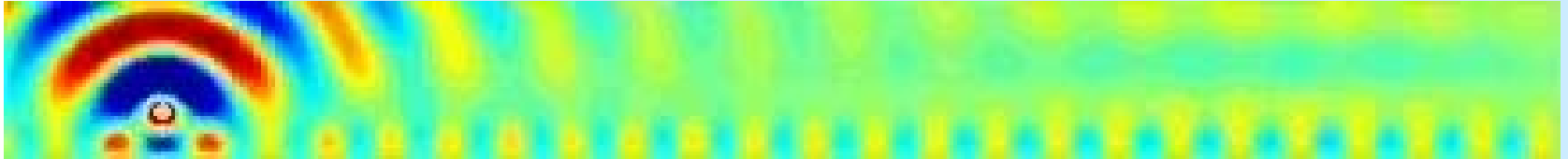
and

$$\tilde{G}(\xi, x_2; x_1^0, x_2^0) = A e^{i\sqrt{k^2 n_0^2 - \xi^2} x_2} + O\left(\frac{1}{|x_2|}\right), \text{ as } x_2 \rightarrow \infty, \quad (2.6)$$

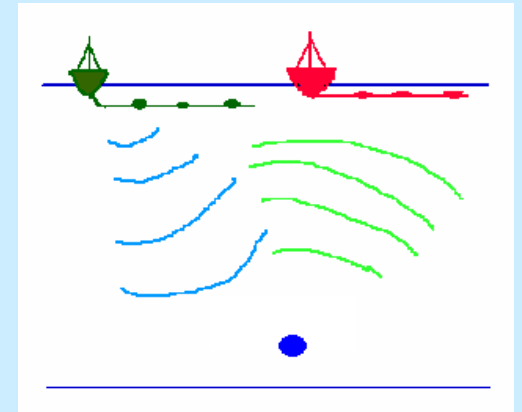
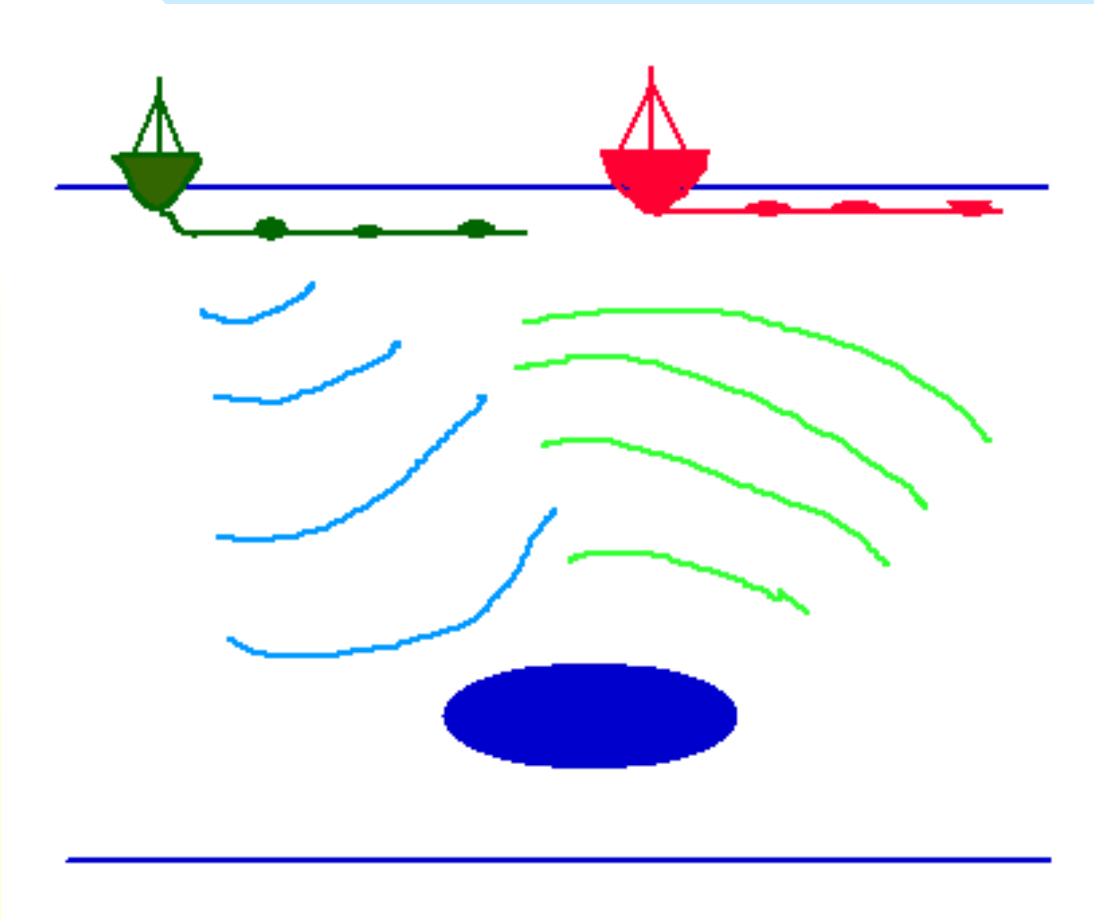
Marine acoustics



Planar Fiber Optic



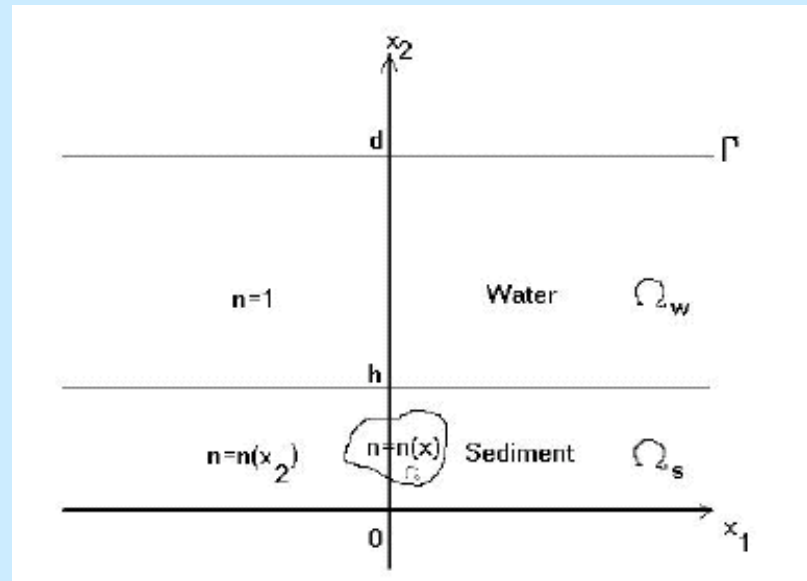
Detecting an object from above



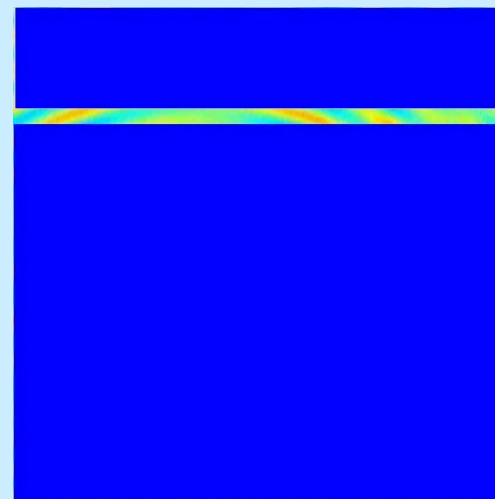
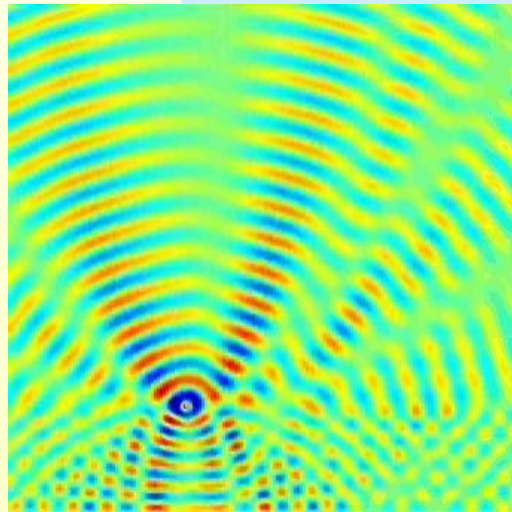
IP1: Given near-field data

$$\Gamma = \{(x_1, x_2) \in R_+^2 \mid x_2 = x_2^0 = \text{constant}\},$$

$$\Gamma_s = \{(x_1^s, x_2^s) \in R_b^2 \mid x_2^s = x_2^{s0} = \text{constant}\}.$$



Given $u^s(x, x_s)$ for $x \in \Gamma$ and $x_s \in \Gamma_s$, find $\partial\Omega$.



Inverse scattering problem ----- a mathematical problem

Let

$$\Gamma = \{(x_1, x_2) \in \mathbb{R}_h^2 \mid x_2 = x_2^0 = \text{constant}\}, \quad (2.13)$$

and

$$\Gamma_s = \{(x_1^s, x_2^s) \in \mathbb{R}_h^2 \mid x_2^s = x_2^{s0} = \text{constant}\} \quad (2.14)$$

The inverse problem we consider is the following:

Given $u^s(x, x^s)$ for $x \in \Gamma$ and $x^s \in \Gamma_s$, find the boundary of the unknown boundary $\partial\Omega$ without knowing which of the three boundary conditions u^s satisfies.

Similar to Colton, Kirsch and Monk, we may prove:

Lemma 3.1. Let u^s be a solution of the Helmholtz equation in the exterior of Ω in \mathbb{R}_h^2 satisfying the boundary conditions (2.9), (2.10) and the outgoing radiation condition (2.11). If $u^s = 0$ on Γ , then $u^s = 0$ in $\mathbb{R}_h^2 \setminus \Omega$.

Lemma 3.2. Let D be a bounded domain such that $\mathbb{R}_h^2 \setminus \bar{D}$ is connected. Assume that k^2 is not an eigenvalue for the bounded domain D with Dirichlet, Neumann or Impedance boundary condition. Let $G(\cdot, x^s)$ be the outgoing Green's function for the waveguide \mathbb{R}_h^2 with source $x^s \in \Gamma_s$. Then the restriction of the set $\{G(\cdot, x^s) : x^s \in \Gamma_s\}$ to ∂D is complete in $L^2(\partial D)$.

Lemma 3.3. The scattering operator U^s corresponding to a scatterer Ω is an injection from $L^{2,\alpha}(\Gamma_s)$ to $L^{2,-\alpha}(\Gamma)$ for $\alpha > 1/2$, if k^2 is neither a Dirichlet, Neumann nor Impedance eigenvalue for Ω .

Lemma 3.4. Let U^s be the scattering operator corresponding to a scatterer Ω , and $G(x, y)$ be the Green's function for the parallel waveguide without any obstacle. Then $G(\cdot, y) \notin \text{Range}(U^s)$ if $y \notin \Omega$.

Theorem 3.5: Assume that k^2 is neither a Dirichlet, Neumann nor Impedance eigenvalue of the direct scattering problem. If Ω_1 and Ω_2 are two scatterers such that for the fixed wave number k^2 the scattered waves $u_1^s(\cdot, x^s)$ and $u_2^s(\cdot, x^s)$, corresponding to the scatterers Ω_1 and Ω_2 respectively, coincide on Γ for all $x^s \in \Gamma_s$. Then

$$\Omega_1 = \Omega_2.$$

Let D be a region containing $\bar{\Omega}$ and $G(x, y)$ the outgoing Green's function with source y . For any fixed $y \in D$, we consider the integral equation

$$\int_{\Gamma_s} u^s(x, x^s) g(x^s; y) dx^s = G(x, y), \text{ for } x \in \Gamma. \quad (4.1)$$

From Lemma 3.3 and Lemma 3.4, we have the following two results:

Theorem

- (1) If $y \in D \setminus \bar{\Omega}$, (4.1) has no solution.
- (2) If (4.1) has a solution $g(x^s; y)$, $x^s \in \Gamma_s$, for given $y \in \Omega$, then the solution is unique if there does not exist an eigenfunction for Ω with homogeneous boundary condition corresponding to the Dirichlet, Neumann or Impedance boundary condition.
- (3) If the equation (4.1) has a solution $g \in L^{2,\alpha}(\Gamma_s)$, $\alpha > 1/2$, for given $y \in \Omega$ and $y \rightarrow x \in \partial\Omega$, then

$$\lim_{y \rightarrow x \in \partial\Omega} \|g(\cdot; y)\|_{L^{2,\alpha}(\Gamma_s)} = \infty. \quad (4.2)$$

Outline

Dual space indicator method with automatic filtering

- (1) Measure the scattered field at $N+1$ points along a straight line (denoted by Γ) for each sound source located on the same line. The measured data is saved in an $(N+1) \times (N+1)$ matrix.
- (2) Choose a searching region that may contain the unknown obstacle (denoted by D). Compute the Green's function $G(X, Y)$ for $N+1$ points $X \in \Gamma$ and each $Y \in D$ approximately (i.e., truncating at a suitable term).
- (3) Solve the regularized linear system

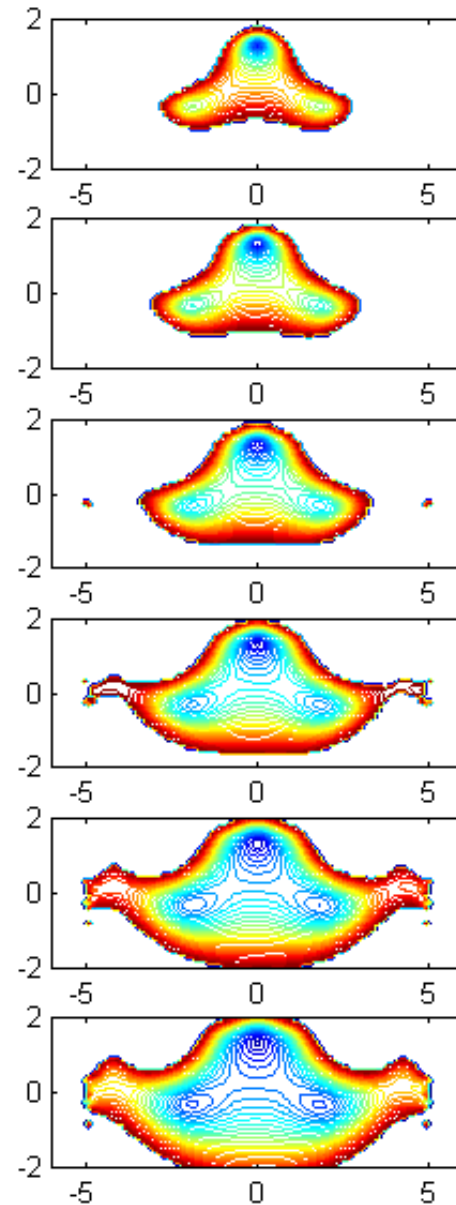
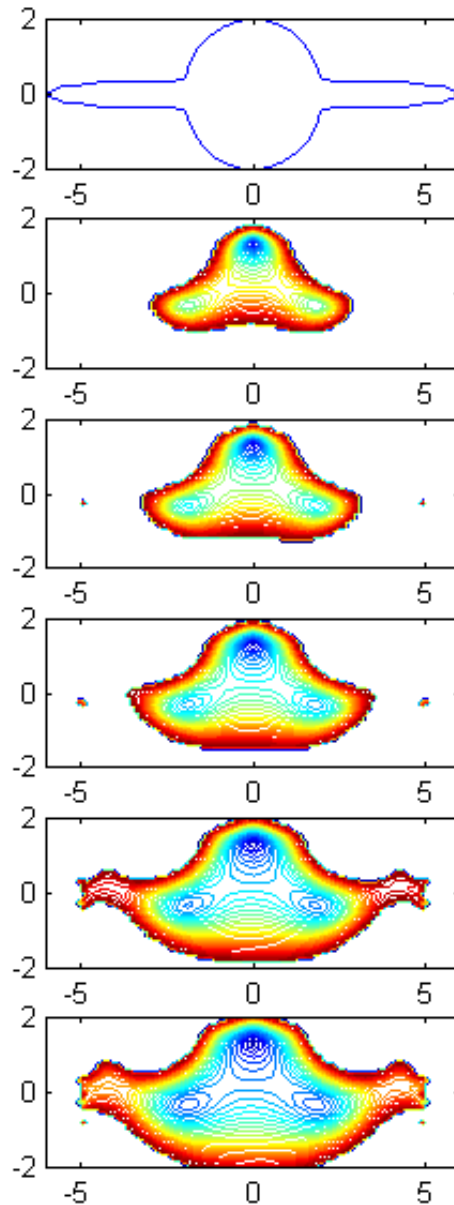
$$(\epsilon I + S^* S)g(\cdot, Y) = S^* G(\cdot, Y), \quad (4.4)$$

for $g(\cdot; Y)$ for each $Y \in D$.

- (4) Compute the norm (L_2 -norm or other norms) $\|g(\cdot, Y)\|$ for each $Y \in D$. Draw the contour of $\|g(\cdot, Y)\|$ as a function of Y on D .
- (5) Use an automatic filtering algorithm to estimate the outline of the unknown target.

Numerical examples

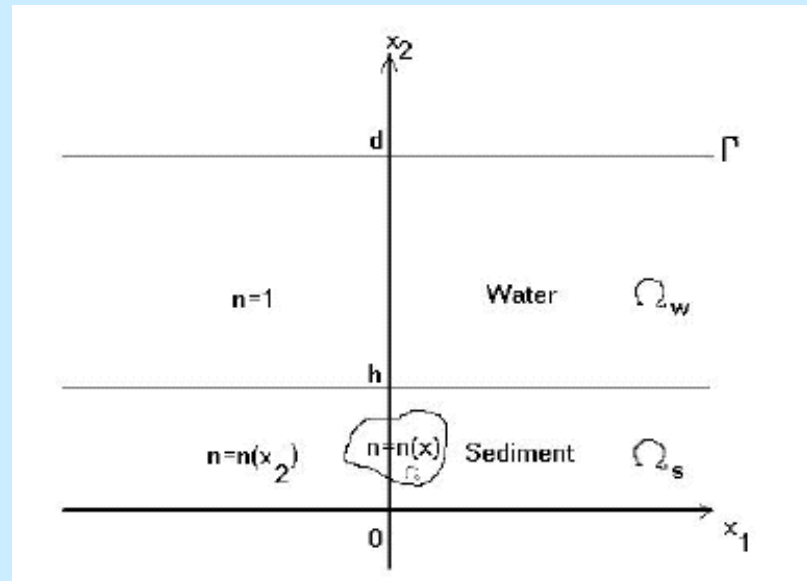
Example 1



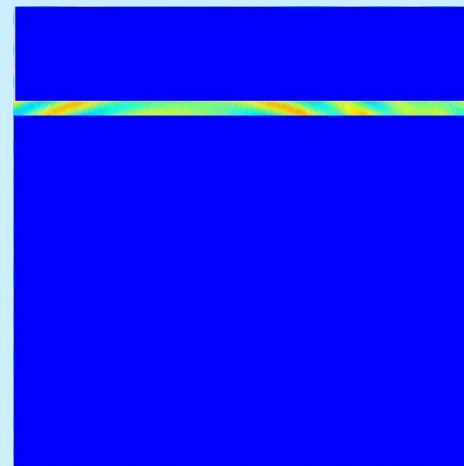
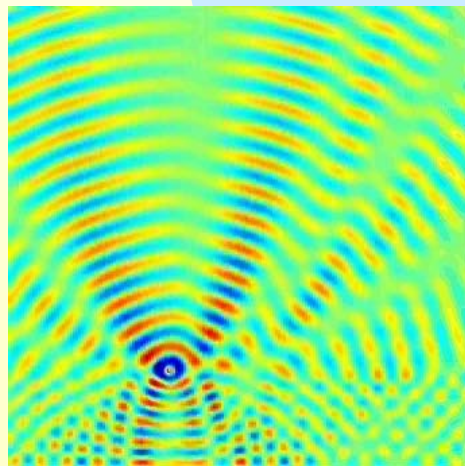
IP2: Given near-field data

$$\Gamma = \{(x_1, x_2) \in \mathbb{R}_+^2 \mid x_2 = x_2^0 = \text{constant}\},$$

$$\Gamma_s = \{(x_1^s, x_2^s) \in \mathbb{R}_h^2 \mid x_2^s = x_2^{s0} = \text{constant}\}.$$



Given $u^s(x, x^s)$ for $x \in \Gamma$ and $x^s \in \Gamma_s$, determine $n(\mathbf{x})$.



Numerical solution of the inverse scattering problem

Recall

$$u(x_1, x_2) + \int_{\Omega} G(\xi_1, \xi_2; x_1, x_2) m(\xi_1, \xi_2) u(\xi_1, \xi_2) d\xi_1 d\xi_2 = G(x_1^0, x_2^0; x_1, x_2).$$

Now we present an numerical example for the inverse problem. We use a regularized Born approximation method to reconstruct the unknown inhomogeneity. Assume that M is small, we have $u(x, z) \simeq G(x_{1s}, x_{2s}; x_1, x_2)$ for $(x_1, x_2) \in \Omega$. Therefore, the scattered field operator

$$F(mG)(x, z) := \int_{\Omega} G(\xi_1, \xi_2; x_1, x_2) \widehat{k}^2 G(\xi_1, \xi_2; x_{1s}, x_{2s}) d\xi_1 d\xi_2, \text{ for } (x_1, x_2) \in \Gamma$$

is the approximation of $F(mu)(x_1, x_2)$ and

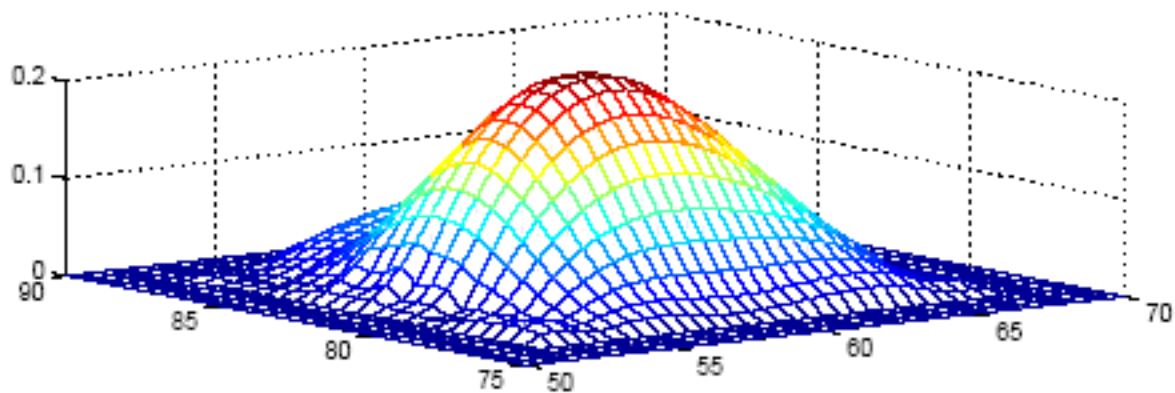
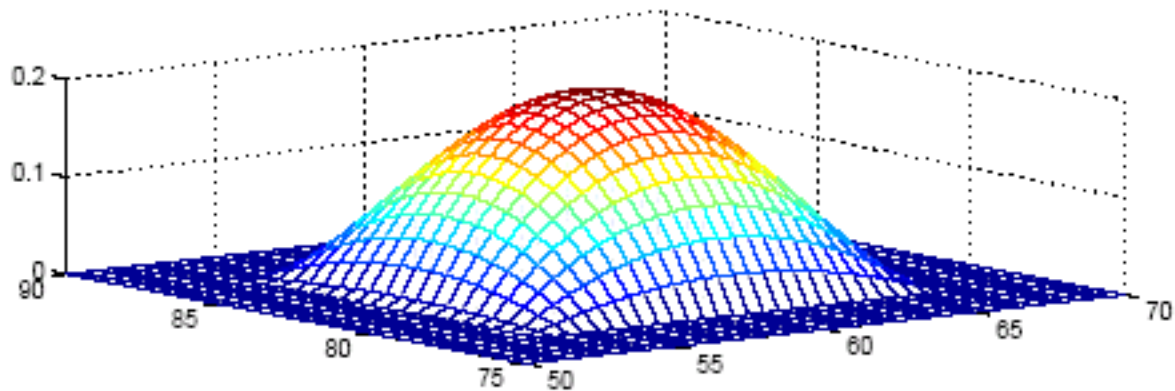
$$F(mG)(x_1, x_2) \simeq G(x_{1s}, x_{2s}; x_1, x_2) - u(x_1, x_2) =: u_*^s(x_1, x_2), \text{ for } (x_1, x_2) \in \Gamma. \quad (4.8)$$

Note that F is a linear operator of m . Discretizing (4.8) we obtain an ill-conditioned linear system $Fm = u_*^s$. The regularized Born approximation gives the system

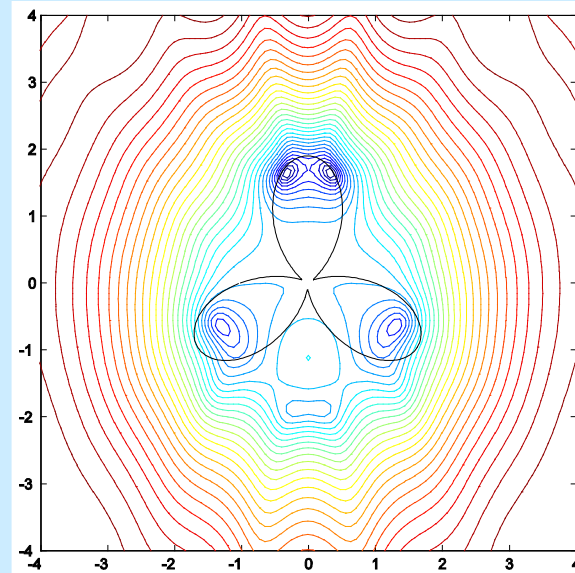
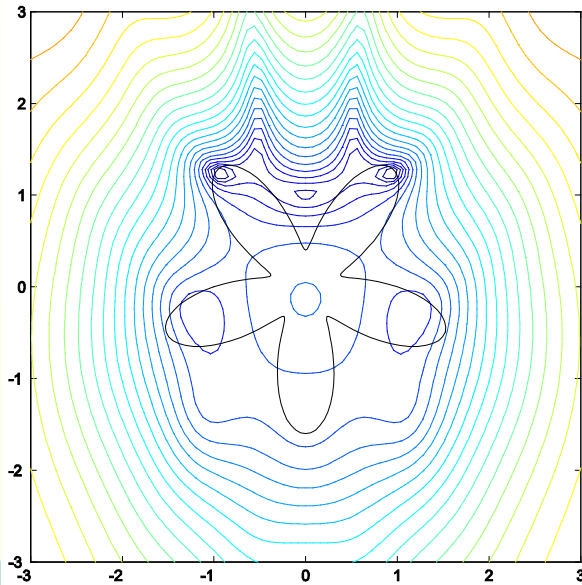
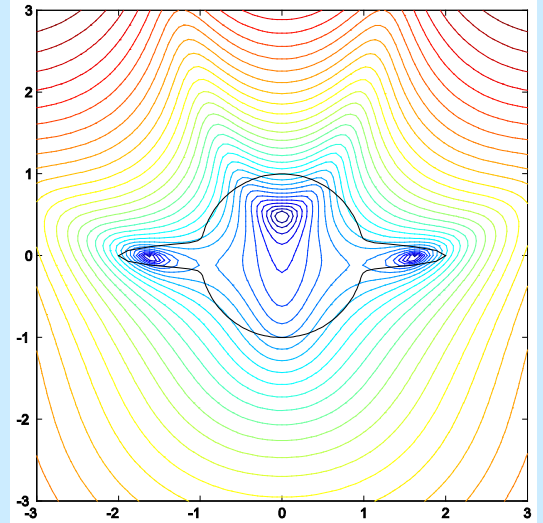
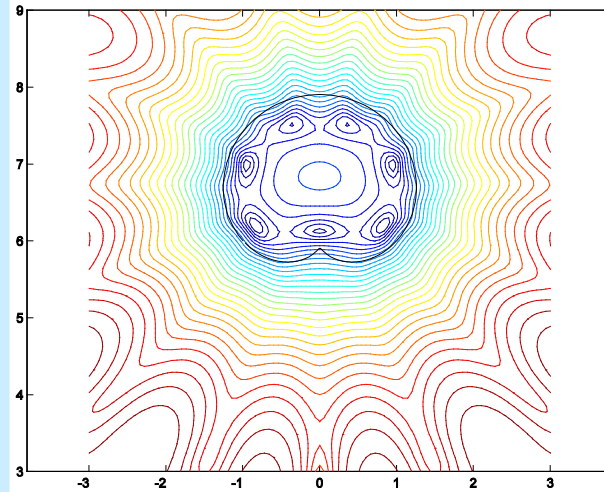
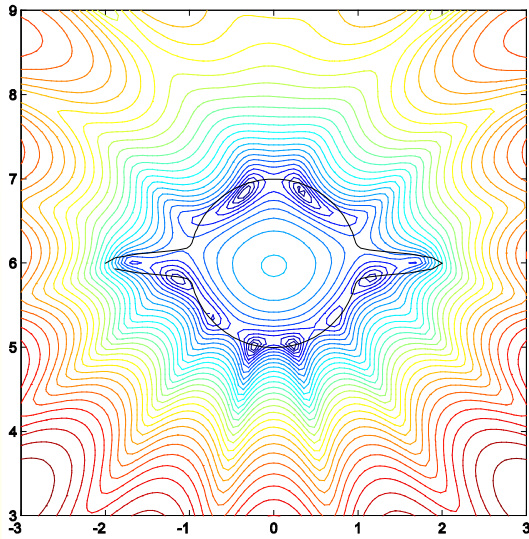
$$(\epsilon I + F^*F)m = F^*u_*^s.$$

Example 1: The inhomogeneity is

$$n^2(x_1, x_2) = \begin{cases} 0.2 \frac{(5^2 - (x - 82.5)^2)(6.6^2 - (x - 60)^2)}{33.3^2} \\ (x, x) \in [53.3, 66.6] \times [77.5, 87.5] \\ 0 \text{ otherwise.} \end{cases}$$



Level Set Method



MARINE ACOUSTICS

Direct and Inverse Problems

This book presents current research trends in the field of underwater acoustic wave direct and inverse problems. Until very recently, little had been published concerning model-based inversions of the bioacoustic and material constants of fish-sized targets located in either the water column or the sediments. This text is the first to investigate inverse problems in an ocean environment with a heavy emphasis placed on the description and resolution of the forward scattering problem.

The ocean-based system is an acoustic waveguide and *Marine Acoustics: Direct and Inverse Problems* is the only book that treats inverse problems in a waveguide. This timely publication addresses many areas of practical interest related to underwater acoustical imaging including ecological survey and clean up, protection of open water harbors, maintenance of offshore petroleum and gas enterprises, as well as other areas of environmental and military concern.

The book is written with several audiences in mind. Engineers and scientists working in ocean acoustics, military scientists interested in submarine detection and long range underwater communications, and geophysicists involved in locating underwater oil fields will find this book of keen interest. The first chapter is an excellent introduction for the reader who is unacquainted with theoretical acoustics and contains all the physics necessary for understanding the book.

James L. Buchanan is Professor of Mathematics at the United States Naval Academy. He has published two books and over 20 articles in function theoretical methods, elasticity, ocean acoustics, inverse problems, and numerical analysis.

Robert P. Gilbert is Goddard Chair of Applied Analysis at the University of Delaware. He is primarily known for his treatise on function theoretical methods applied to partial differential equations and his work in elasticity. He has published 13 books, over 200 articles, and is founding editor of two mathematics journals, *Complex Variables and Applications* and *Applied Analysis*.

Armand Wirgin is Directeur de Recherche at the Laboratoire de Mécanique et d'Acoustique of the Centre National de la Recherche Scientifique in France. He has published over 170 articles, book chapters, and books on direct and inverse scattering of electromagnetic and elastic waves.

Yongzhi S. Xu is a UC Foundation Professor of Mathematics at University of Tennessee at Chattanooga. He has published over 60 papers on direct and inverse problems in ocean acoustics, function theoretical methods, mathematical study of carotoma and cataportans, and other applied partial differential equation problems.

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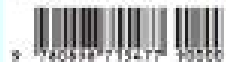
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