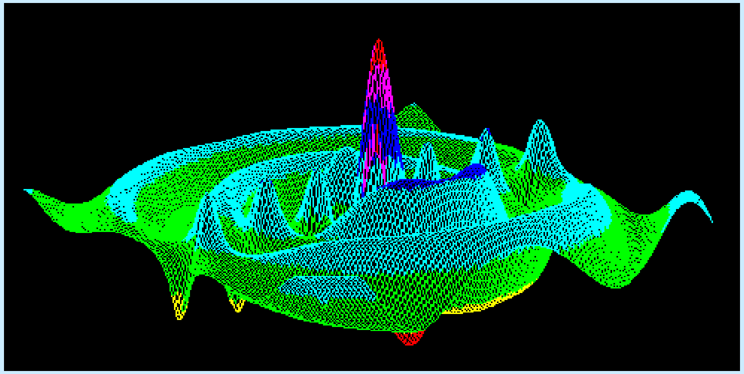
Detecting Objects in a layered medium



Yongzhi Steve Xu
Department of Mathematics
University of Louisville

1. The Mathematical model

Let $R^2_+ = \{\mathrm{x} \in R^2 | x_2 \geq 0\}$, where $\mathrm{x} = (x_1, x_2)$. $n(\mathrm{x})$ is defined on R^2_+ with the following properties:

Let

$$n_0(x_2) = \begin{cases} n_0, & for\ 0 < x_2 < h, \\ 1, & for\ h < x_2 < \infty, \end{cases} \tag{1.1}$$

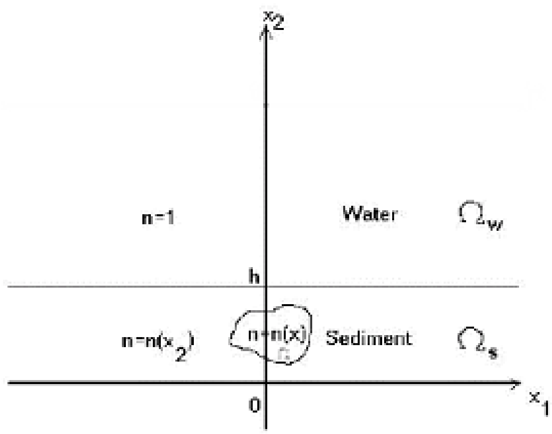
where $n_0 > 1$ is a constant. We assume that the inhomogeneity is contained in a bounded domain $\Omega \in R^2_h = \{\mathrm{x} \in R^2 | 0 < x_2 < h\}$ with C^2 boundary having outward normal vector ν.

$$n(\mathrm{x}) = n_0(x_2), \text{ for } \mathrm{x} \notin \Omega. \tag{1.2}$$

We consider the following problem: given point sourse or plane wave incident wave u^i satisfying

$$\triangle u^i + k^2 n_0^2(x_2) u^i = f(\mathrm{x})\ ,\ in\ R^2_+, \tag{1.3}$$

find the total field $u = u^s + u^i$ such that

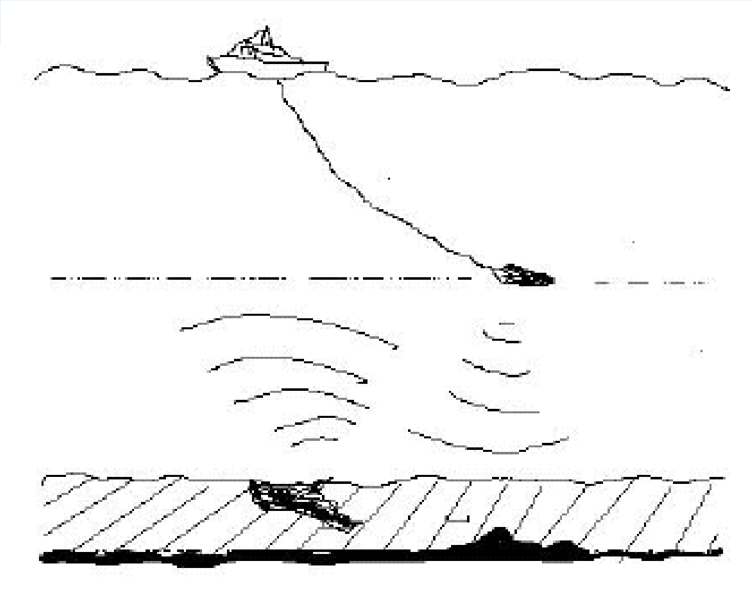


$$\triangle u^s + k^2 n^2(\mathrm{x}) u^s = 0,\ ,\ in\ R^2_+, \tag{1.4}$$

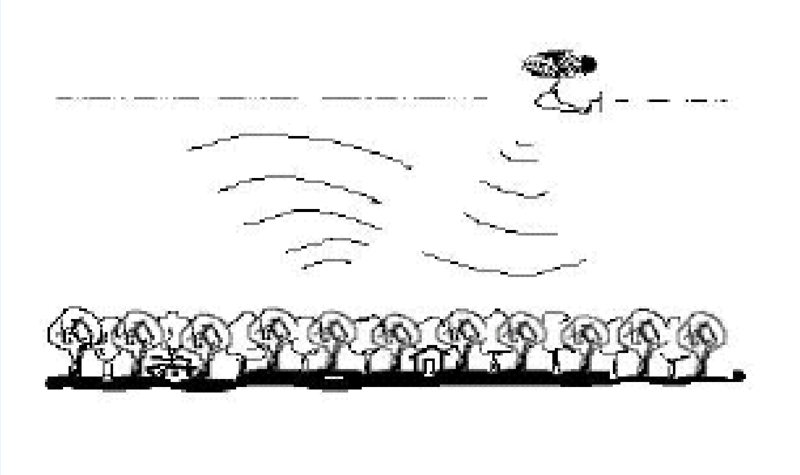
$$u(\mathrm{x}) = u^s(\mathrm{x}) + u^i(\mathrm{x}) = 0\ ,\ when\ x_2 = 0, \tag{1.5}$$

$$\rho_0 u^+(\mathbf{x}) = \rho u^-(\mathbf{x}), \quad \frac{\partial u^+(\mathbf{x})}{\partial \nu} = \frac{\partial u^-(\mathbf{x})}{\partial \nu}, \text{ on } \partial\Omega. \tag{1.6}$$

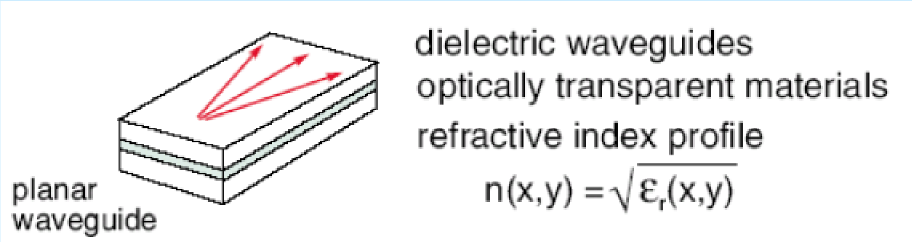
Sound wave

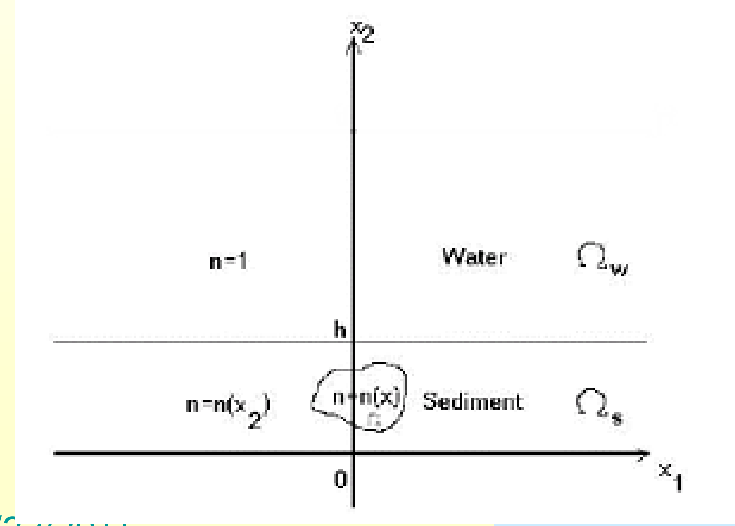


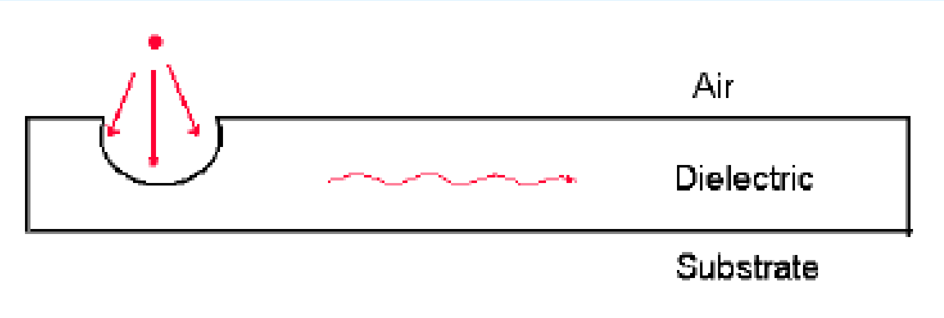
Electric magnetic wave



Light wave







2.1: Green's function and its far-field pattern

A function $G(\,\cdot\,; x_1^0, x_2^0)$ is the outgoing Green's function from the source at $\mathbf{x^0} = (x_1^0, x_2^0)$ for the time-harmonic wave in a layered half-space, if $G(x_1, x_2; x_1^0, x_2^0)$ satisfies

$$\triangle G + k^2 n_0^2(x_2) G = -\delta(|\mathbf{x} - \mathbf{x^0}|)\ ,\ in\ R_+^2 \qquad (2.1)$$

in the generalized function sense;

$$G(x_1, 0; x_1^0, x_2^0) = 0, \qquad (2.2)$$

$$G(x_1, x_2; x_1^0, x_2^0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(\xi, x_2; x_1^0, x_2^0) e^{i\xi|x_1 - x_1^0|} d\xi, \qquad (2.3)$$

where $\tilde{G}(\xi, x_2; x_1^0, x_2^0)$ satisfies

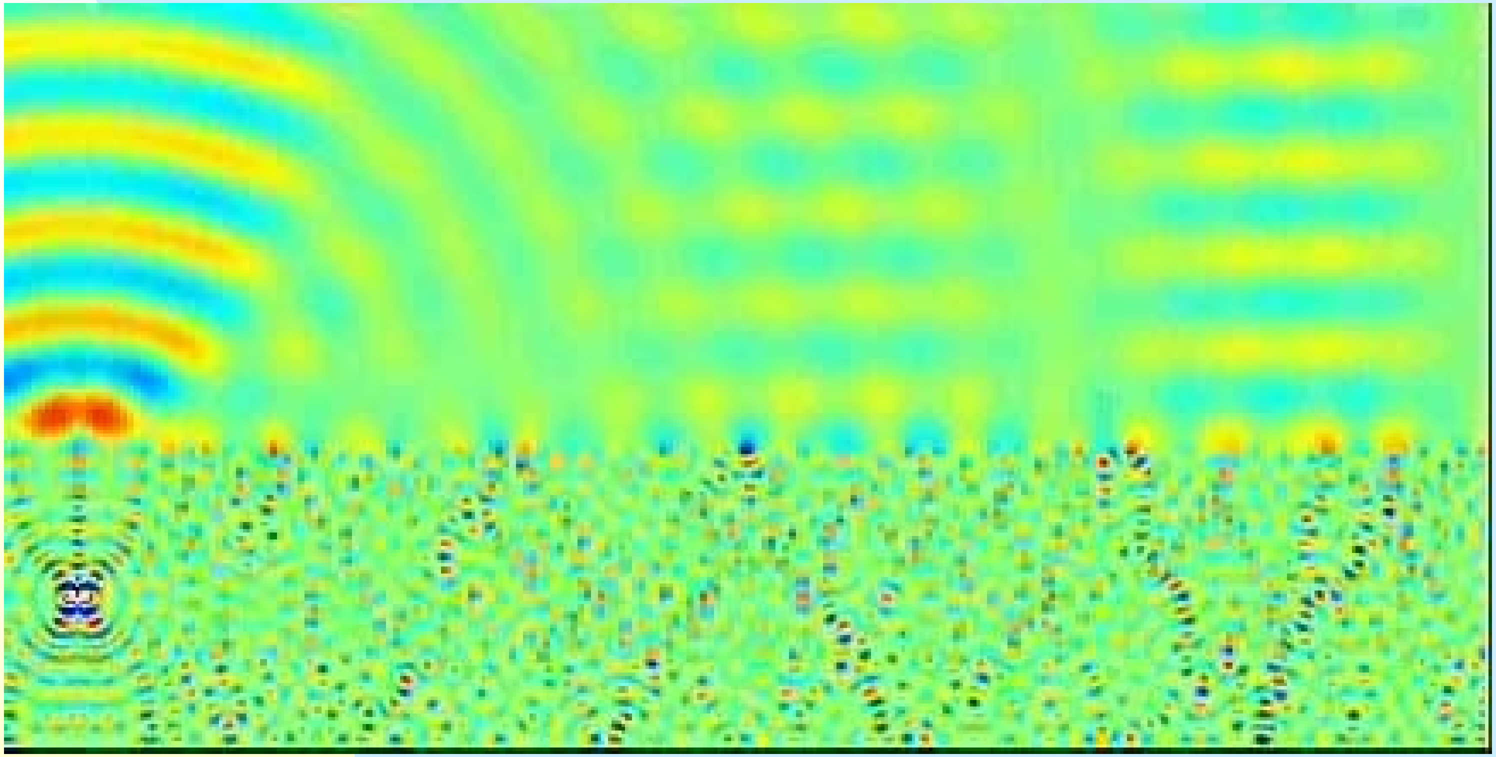
$$\tilde{G}''(x_2) + [k^2 n_0^2(x_2) - \xi^2]\tilde{G}(x_2) = -\delta(x_2 - x_2^0),\ 0 < x_2 < \infty, \qquad (2.4)$$

$$\tilde{G}(\xi, 0; x_1^0, x_2^0) = 0, \qquad (2.5)$$

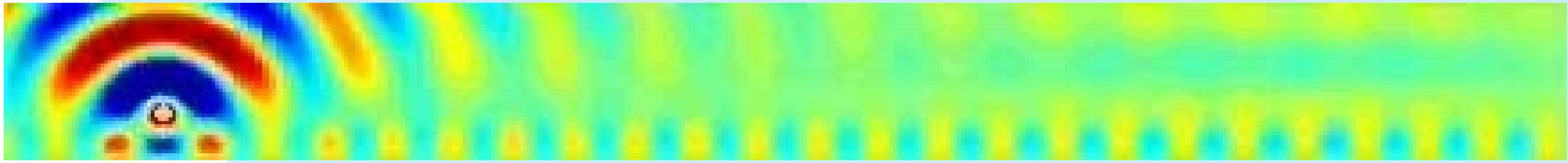
and

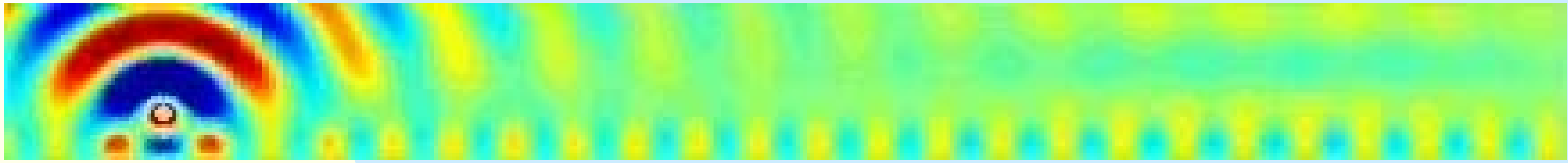
$$\tilde{G}(\xi, x_2; x_1^0, x_2^0) = A e^{i\sqrt{k^2 n_0^2 - \xi^2} x_2} + O\left(\frac{1}{|x_2|}\right),\ as\ x_2 \to \infty, \qquad (2.6)$$

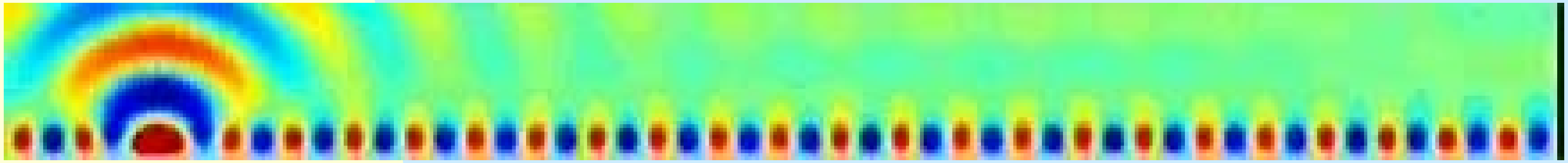
Marine acoustics



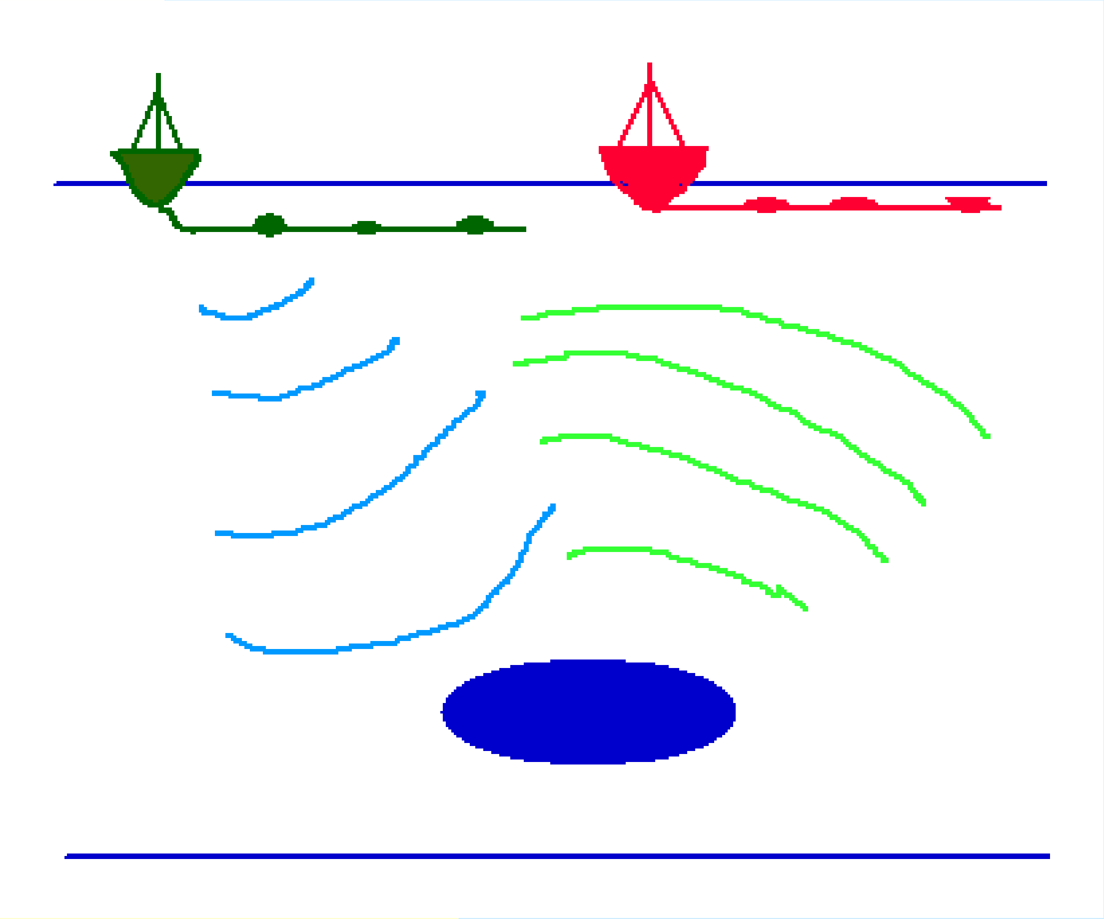
Planar Fiber Optic

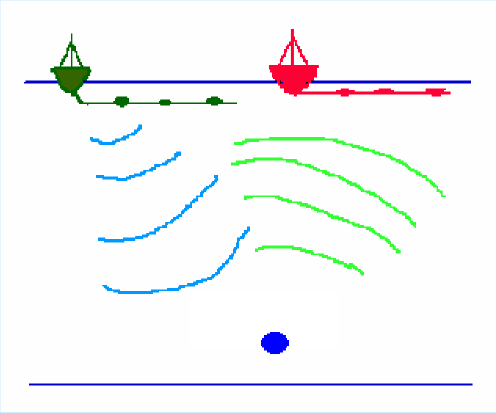






Detecting an object from above

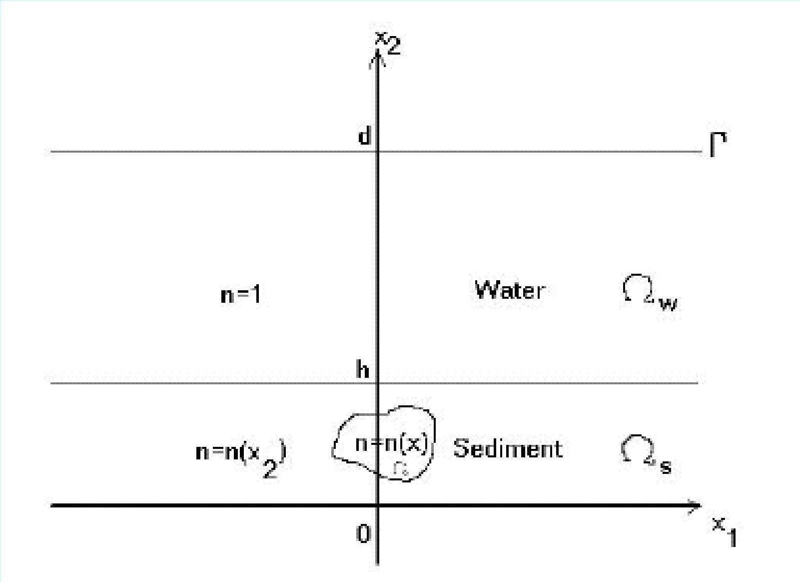




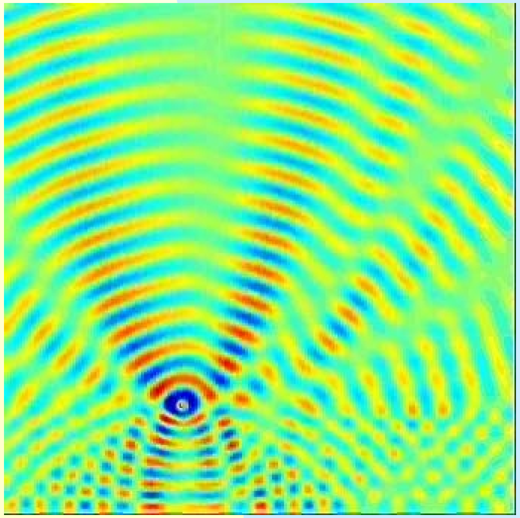
IP1: Given near-field data

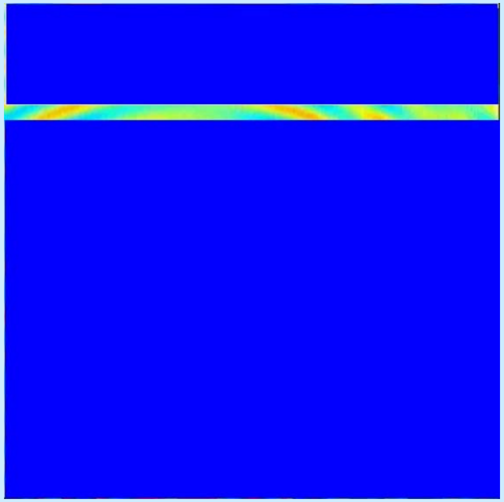
$$\Gamma = \{(x_1, x_2) \in R^2_+ | x_2 = x_2^0 = \text{constant}\},$$

$$\Gamma_s = \{(x_1^s, x_2^s) \in R^2_h | x_2^s = x_2^{s0} = \text{constant}\}.$$



Given $u^s(x, x_s)$ **for** $x \in \Gamma$ **and** $x_s \in \Gamma_s$**, find** $\partial\Omega$**.**





Inverse scattering problem
------- a mathematical problem

Let

$$\Gamma = \{(x_1, x_2) \in \mathbb{R}_h^2 | x_2 = x_2^0 = \text{constant}\}, \tag{2.13}$$

and

$$\Gamma_s = \{(x_1^s, x_2^s) \in \mathbb{R}_h^2 | x_2^s = x_2^{s0} = \text{constant}\} \tag{2.14}$$

The inverse problem we consider is the following:

Given $u^s(x, x^s)$ for $x \in \Gamma$ and $x^s \in \Gamma_s$, find the boundary of the unknown boundary $\partial\Omega$ without knowing which of the three boundary conditions u^s satisfies.

Mathematical Theory

Similar to Colton, Kirsch and Monk, we may prove:

Lemma 3.1. Let u^s be a solution of the Helmholtz equation in the exterior of Ω in $\mathbb{R}^2_h$ satisfying the boundary conditions (2.9), (2.10) and the outgoing radiation condition (2.11). If $u^s = 0$ on Γ, then $u^s = 0$ in $\mathbb{R}^2_h \setminus \Omega$.

Lemma 3.2. Let D be a bounded domain such that $\mathbb{R}^2_h \setminus \bar{D}$ is connected. Assume that k^2 is not an eigenvalue for the bounded domain D with Dirichlet, Neumann or Impedance boundary condition. Let $G(\cdot, x^s)$ be the outgoing Green's function for the waveguide $\mathbb{R}^2_h$ with source $x^s \in \Gamma_s$. Then the restriction of the set $\{G(\cdot, x^s) : x^s \in \Gamma_s\}$ to ∂D is complete in $L^2(\partial D)$.

Lemma 3.3. The scattering operator U^s corresponding to a scatterer Ω is an injection from $L^{2,\alpha}(\Gamma_s)$ to $L^{2,-\alpha}(\Gamma)$ for $\alpha > 1/2$, if k^2 is neither a Dirichlet, Neumann nor Impedance eigenvalue for Ω.

Lemma 3.4. Let U^s be the scattering operator corresponding to a scatterer Ω, and $G(x, y)$ be the Green's function for the parallel waveguide without any obstacle. Then $G(\cdot, y) \notin Range(U^s)$ if $y \notin \Omega$.

Theorem 3.5: Assume that k^2 is neither a Dirichlet, Neumann nor Impedance eigenvalue of the direct scattering problem. If Ω_1 and Ω_2 are two scatterers such that for the fixed wave number k^2 the scattered waves $u_1^s(\cdot, x^s)$ and $u_2^s(\cdot, x^s)$, corresponding to the scatterers Ω_1 and Ω_2 respectively, coincide on Γ for all $x^s \in \Gamma_s$. Then

$$\Omega_1 = \Omega_2.$$

Generalized dual space

Indicator method

Let D be a region containing $\overline{\Omega}$ and $G(x, y)$ the outgoing Green's function with source y. For any fixed $y \in D$, we consider the integral equation

$$\int_{\Gamma_s} u^s(x, x^s) g(x^s; y)\, dx^s = G(x, y), \quad \text{for } x \in \Gamma. \tag{4.1}$$

From Lemma 3.3 and Lemma 3.4, we have the following two results:

Theorem

(1) If $y \in D \setminus \bar{\Omega}$, (4.1) has no solution.

(2) If (4.1) has a solution $g(x^s; y)$, $x^s \in \Gamma_s$, for given $y \in \Omega$, then the solution is unique if there does not exist an eigenfunction for Ω with homogeneous boundary condition corresponding to the Dirichlet, Neumann or Impedance boundary condition.

(3) If the equation (4.1) has a solution $g \in L^{2,\alpha}(\Gamma_s)$, $\alpha > 1/2$, for given $y \in \Omega$ and $y \to x \in \partial\Omega$, then

$$\lim_{y \to x \in \partial\Omega} \|g(\cdot; y)\|_{L^{2,\alpha}(\Gamma_s)} = \infty. \tag{4.2}$$

Outline

Dual space indicator method with automatic filtering

(1) Measure the scattered field at $N+1$ points along a straight line (denoted by Γ) for each sound source located on the same line. The measured data is saved in an $(N+1)\times(N+1)$ matrix.

(2) Choose a searching region that may contain the unknown obstacle (denoted by D). Compute the Green's function $G(X,Y)$ for $N+1$ points $X \in \Gamma$ and each $Y \in D$ approximately (i.e., truncating at a suitable term).

(3) Solve the regularized linear system

$$(\epsilon I + S^*S)g(\cdot,Y) = S^*G(\cdot,Y), \tag{4.4}$$

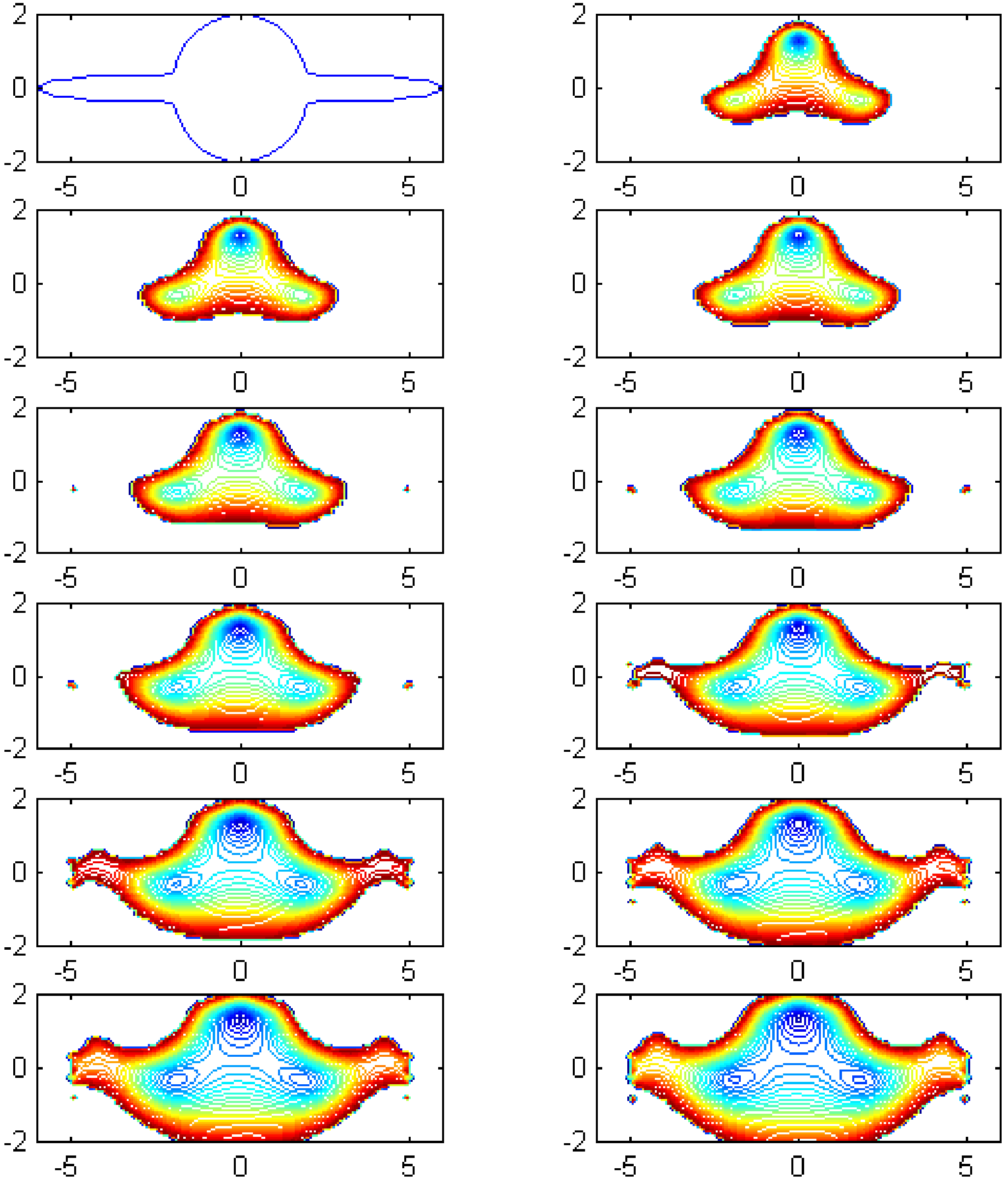
for $g(\cdot;Y)$ for each $Y \in D$.

(4) Compute the norm (L_2-norm or other norms) $\|g(\cdot,Y)\|$ for each $Y \in D$. Draw the contour of $\|g(\cdot,Y)\|$ as a function of Y on D.

(5) Use an automatic filtering algorithm to estimate the outline of the unknown target.

Numerical examples

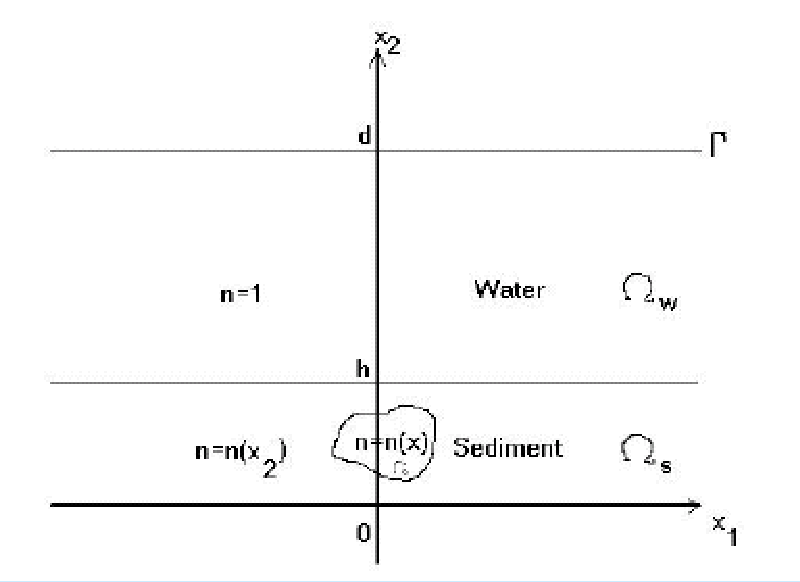
Example 1



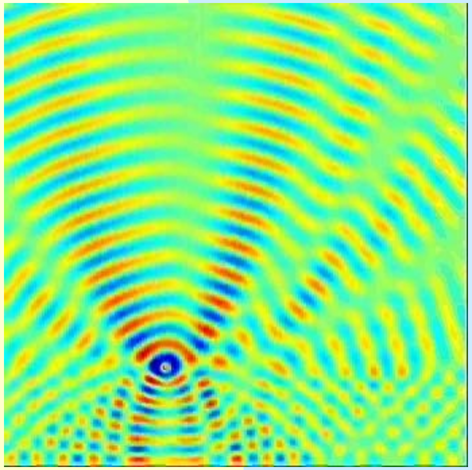
IP2: Given near-field data

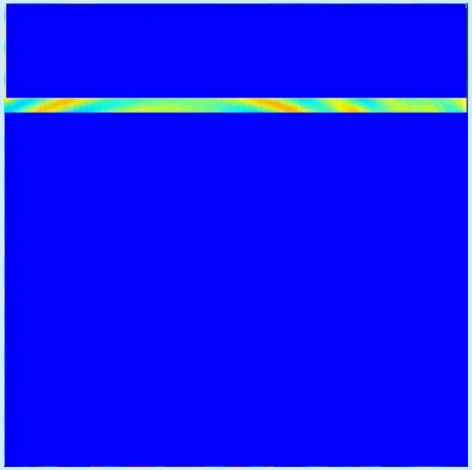
$$\Gamma = \{(x_1, x_2) \in R^2_+ | x_2 = x_2^0 = \text{constant}\},$$

$$\Gamma_s = \{(x_1^s, x_2^s) \in R^2_h | x_2^s = x_2^{s0} = \text{constant}\}.$$



Given $u^s(x, x^s)$ for $x \in \Gamma$ and $x^s \in \Gamma_s$, determine $n(\mathrm{x})$.





Numerical solution of the inverse scattering problem

Recall

$$u\left(x_1, x_2\right) + \int_\Omega G\left(\xi_1, \xi_2; x_1, x_2\right) \mathrm{m}(\xi_1, \xi_2) \mathrm{u}\left(\xi_1, \xi_2\right) \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 = \mathrm{G}\left(\mathrm{x}_1^0, \mathrm{x}_2^0; \mathrm{x}_1, \mathrm{x}_2\right).$$

Now we present an numerical example for the inverse problem. We use a regularized Born approximation method to reconstruct the unknown inhomogeniety. Assume that M is small, we have $u\left(x, z\right) \simeq G\left(x_{1s}, x_{2s}; x_1, x_2\right)$ for $\left(x_1, x_2\right) \in \Omega$. Therefore, the scattered field operator

$$F(mG)(x, z) := \int_\Omega G\left(\xi_1, \xi_2; x_1, x_2\right) \widehat{\mathrm{k}^2} \mathrm{G}\left(\xi_1, \xi_2; \mathrm{x}_{1s}, \mathrm{x}_{2s}\right) \mathrm{d}\xi_1 \, \mathrm{d}\xi_2, \text{ for } (\mathrm{x}_1, \mathrm{x}_2) \in \Gamma$$

is the approximation of $F(mu)(x_1, x_2)$ and

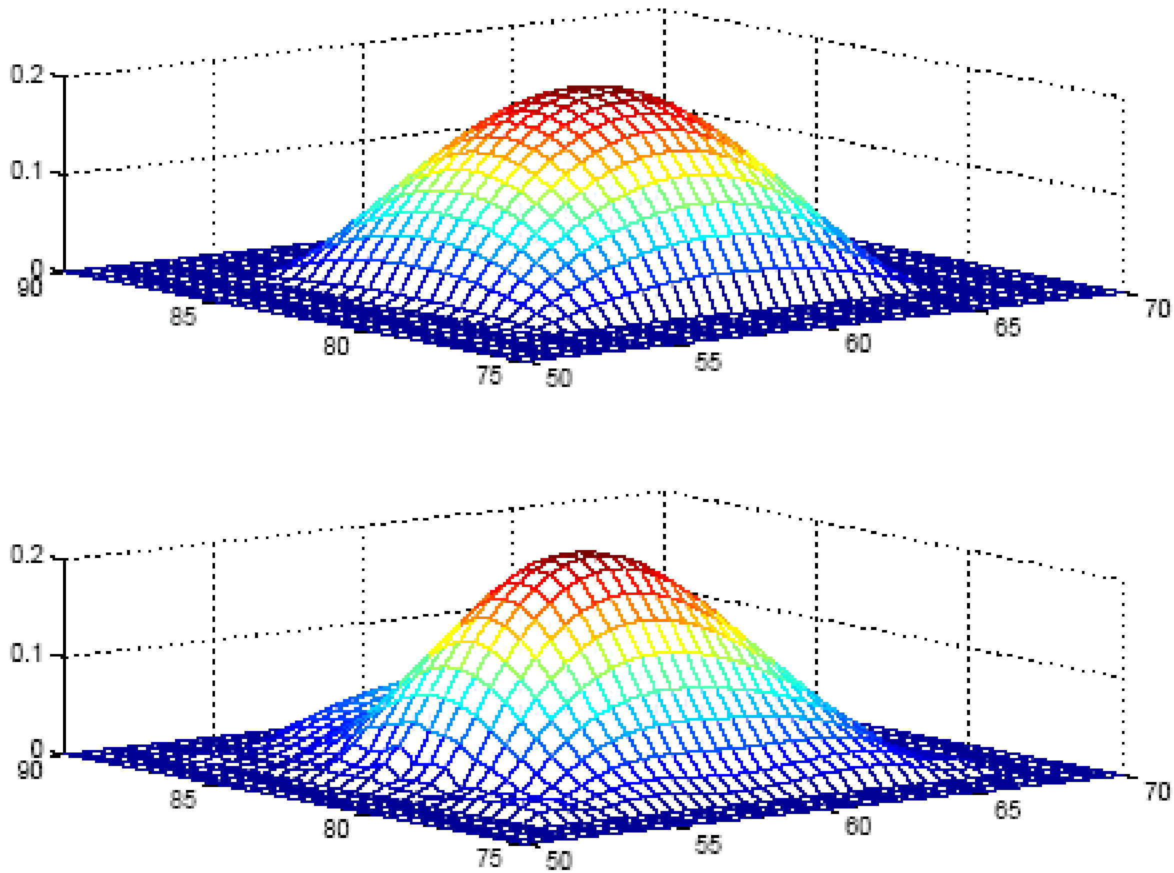
$$F(mG)(x_1, x_2) \simeq G\left(x_{1s}, x_{2s}; x_1, x_2\right) - u(x_1, x_2) =: u_*^s(x_1, x_2), \; for \; (x_1, x_2) \in \Gamma. \qquad (4.8)$$

Note that F is a linear operator of m. Discretizing (4.8) we obtain an ill-conditioned linear system $\mathbf{F}m = u_*^s$. The regularized Born approximation gives the system

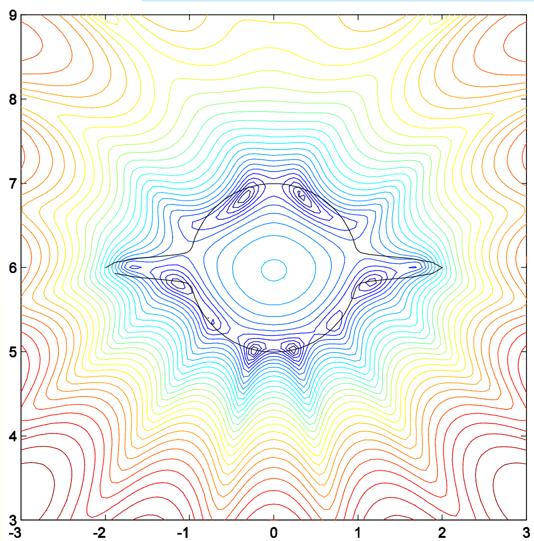
$$(\epsilon I + \mathbf{F}^* \mathbf{F}) m = \mathbf{F}^* u^*.$$

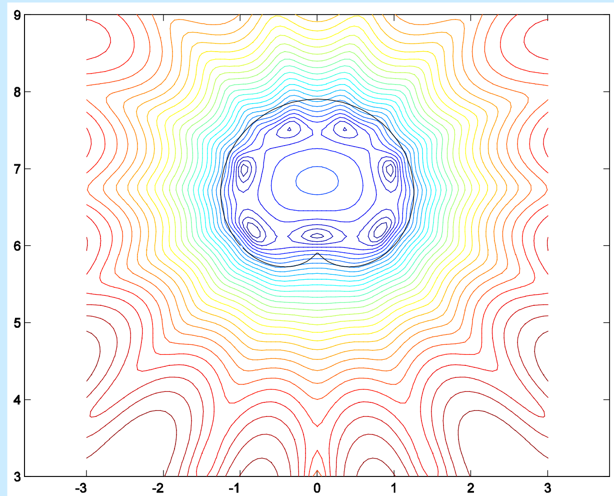
Example 1: The inhomogeniety is

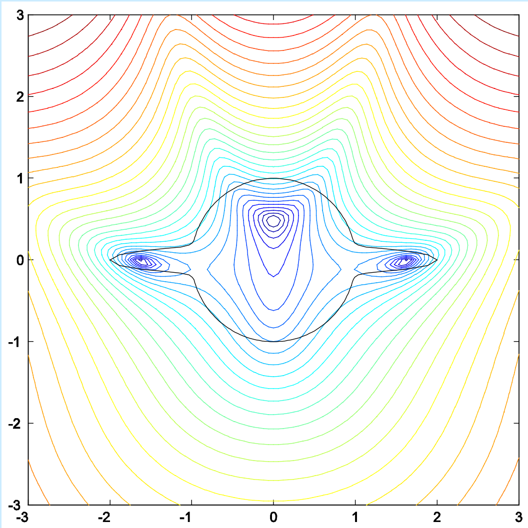
$$n^2(x_1, x_2) = \begin{cases} 0.2\frac{(5^2-(z-82.5)^2)(6.6^2-(x-60)^2)}{33.3^2} \\ (x, z) \in [53.3, 66.6] \times [77.5, 87.5] \\ 0 \quad otherwise. \end{cases}$$

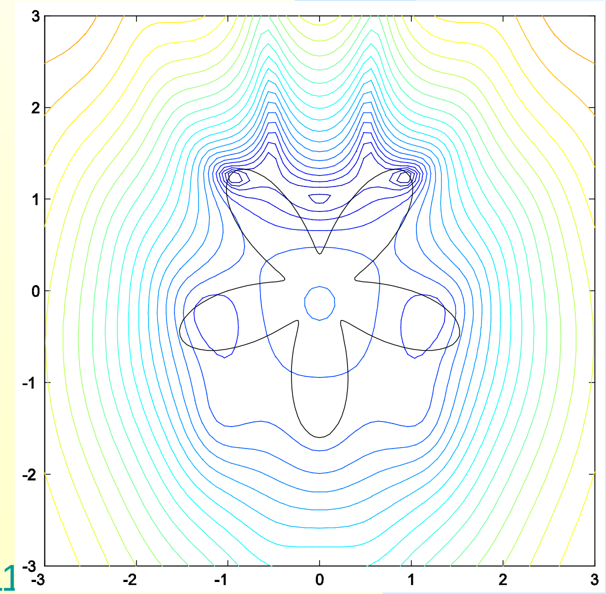


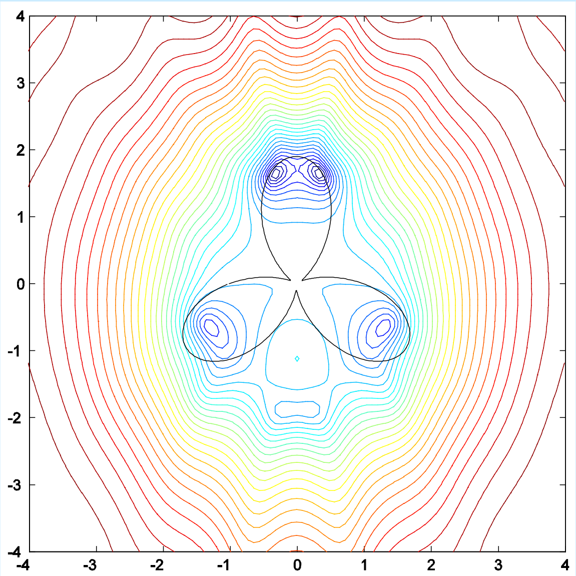
Level Set Method

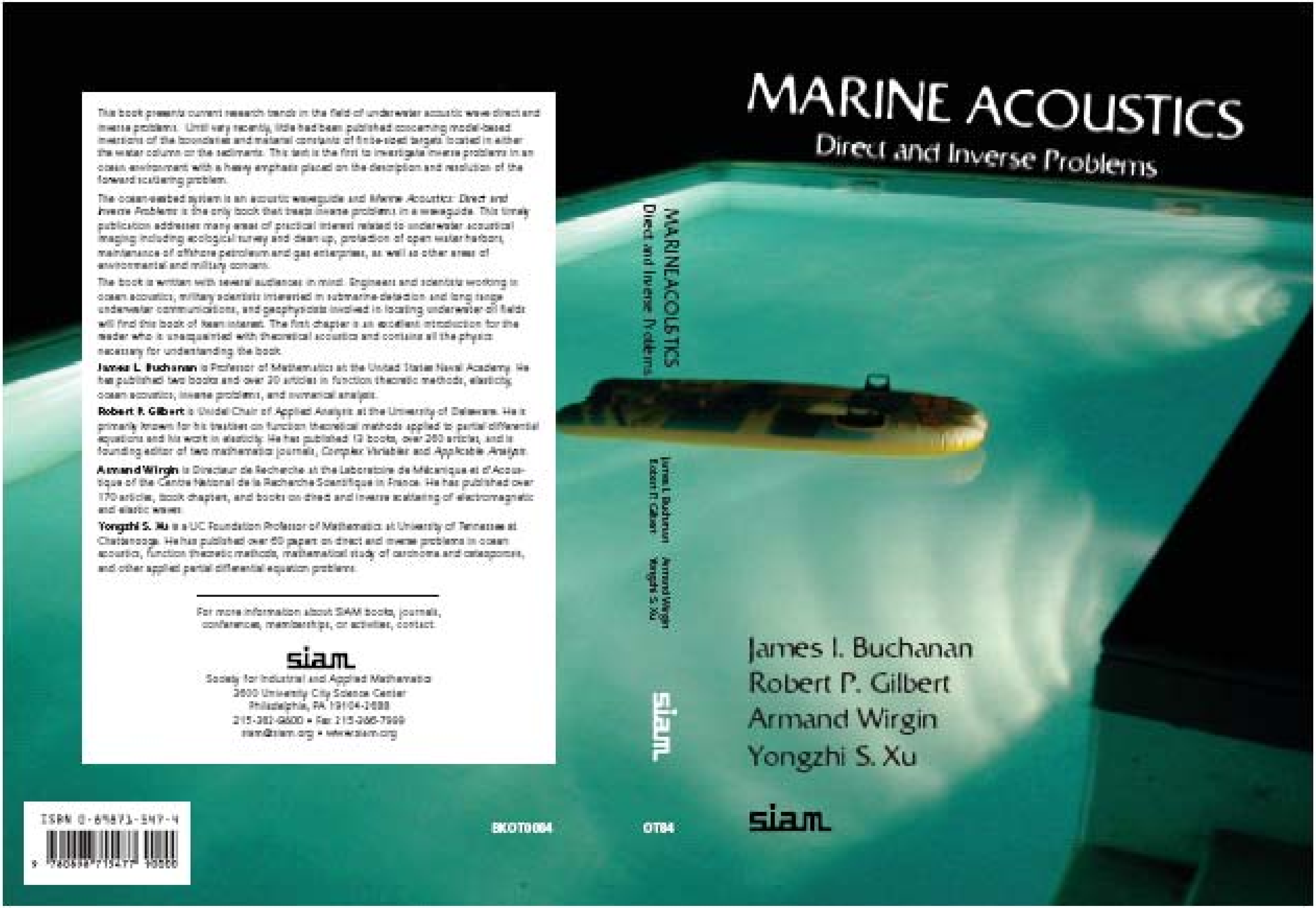











MARINE ACOUSTICS
Direct and Inverse Problems
James L. Buchanan
Robert P. Gilbert
Armand Wirgin
Yongzhi S. Xu
siam