(3, r)-regular graphs

Melanie Laffin
(Joint Work with Sibel Özkan)

Department of Mathematical Sciences
Michigan Technological University
Houghton, MI 49931, USA

12 May 2011
Definition $(t, r)$-regular graph

A simple, undirected graph is $(t, r)$-regular if the cardinality of the open neighborhood set of every $t$ independent vertices is $r$. 
**Definition**

**Definition** ($(t, r)$-regular graph)

A simple, undirected graph is $(t, r)$-regular if the cardinality of the open neighborhood set of every $t$ independent vertices is $r$.

Examples:
Definition \((t, r)\)-regular graph

A simple, undirected graph is \((t, r)\)-regular if the cardinality of the open neighborhood set of every \(t\) independent vertices is \(r\).

Examples:

\[
\begin{array}{c}
\quad \\
\end{array}
\]

(2,2)-regular
Definition \((t, r)\)-regular graph

A simple, undirected graph is \((t, r)\)-regular if the cardinality of the open neighborhood set of every \(t\) independent vertices is \(r\).

Examples:

\begin{itemize}
  \item \((2, 2)\)-regular
  \item \((3, r)\)-regular graphs
\end{itemize}
Definition (\((t, r)\)-regular graph)

A simple, undirected graph is \((t, r)\)-regular if the cardinality of the open neighborhood set of every \(t\) independent vertices is \(r\).

Examples:

\[
\begin{array}{c}
\text{(2,2)-regular} \\
\end{array}
\]

\[
\begin{array}{c}
\text{(t, 1)-regular} \\
\end{array}
\]

Caution! \((t, r)\)-regular graphs *may* refer to those graphs where any \(t\) vertices have \(r\) neighbors.
Can we determine the structure of \((t, r)\)-regular graphs?
Yes!
Can we determine the structure of $(t, r)$-regular graphs?
Yes! Well, mostly. If the order is large enough.
Can we determine the structure of \((t, r)\)-regular graphs?
Yes! Well, mostly. If the order is large enough.

Can we quantify “large enough”?
Yes!
Can we determine the structure of \((t, r)\)-regular graphs?
Yes! Well, mostly. If the order is large enough.

Can we quantify “large enough”?
Yes! (But can we do better?)
(1, r)-regular and (2, r)-regular graphs are (1, r)-regular graphs are r-regular graphs.
(1, \(r\))-regular and (2, \(r\))-regular graphs are \(r\)-regular graphs.

(2, \(r\))-regular graphs of “large enough” order were characterized by Faudree and Knisley (1996).
(1, r)-regular and (2, r)-regular

(1, r)-regular graphs are r-regular graphs. (2, r)-regular graphs of “large enough” order were characterized by Faudree and Knisley (1996).

Theorem ([1])

If r, s, p are nonnegative integers and G is a (2, r)-regular graph of order sufficiently large n, G is isomorphic to $K_s \vee mK_p$ where $2(p - 1) + s = r$. 
(t, r)-regular graphs for \( t > 2 \)

Jamison and Johnson showed that Faudree-Knisley form does not hold for \( t > 2 \).
(t, r)-regular graphs for \( t > 2 \)

Jamison and Johnson showed that Faudree-Knisley form does not hold for \( t > 2 \).
However, if \( t > 2 \): \( G \) is “almost” the join of \( mK_p \) with a graph \( H \) which is “almost” a clique.
Jamison and Johnson showed that Faudree-Knisley form does not hold for $t > 2$.

However, if $t > 2$: $G$ is “almost” the join of $mK_p$ with a graph $H$ which is “almost” a clique for $m \geq t$ and $p$ such that $t(p - 1) + n(H) = r$ where $n(H)$ denotes the number of vertices in $H$. 
What is sufficiently large?

We know what “large” \((t, r)\)-regular graphs look like...

Theorem ([3])

Suppose that \(r \geq 1\) and \(G\) is a \((2, r)\)-regular graph on \(n\) vertices. If \(n \geq N(2, r)\), where

\[
N(2, 1) = 4,
N(2, 2) = 6,
N(2, 3) = 8,
\text{ and } N(2, r) = (r - 1)^2 + 2
\]

for \(r \geq 4\), then \(G = K_s \lor mK_p\) for some integers \(s \geq 0\), \(m \geq 2\), \(p \geq 1\) satisfying \(n = s + mp\) and \(r = s + 2(p - 1)\)
What is sufficiently large?

We know what “large” \((t, r)\)-regular graphs look like... but what the heck does large mean?

Theorem ([3])

Suppose that \(r \geq 1\) and \(G\) is a \((2, r)\)-regular graph on \(n\) vertices. If \(n \geq N(2, r)\), where:

- \(N(2, 1) = 4\),
- \(N(2, 2) = 6\),
- \(N(2, 3) = 8\),
- \(N(2, r) = (r - 1)^2 + 2\) for \(r \geq 4\),

then \(G = K_s \lor mK_p\) for some integers \(s \geq 0\), \(m \geq 2\), \(p \geq 1\) satisfying \(n = s + mp\) and \(r = s + 2(p - 1)\).
What is sufficiently large?

We know what “large” \((t, r)\)-regular graphs look like... but what the heck does large mean? Johnson and Morgan quantified “sufficiently large” for \((2, r)\)-regular graphs.

\[\text{Theorem} \quad \text{(3)}\]

Suppose that \(r \geq 1\) and \(G\) is a \((2, r)\)-regular graph on \(n\) vertices. If \(n \geq N(2, r)\), where \(N(2, 1) = 4\), \(N(2, 2) = 6\), \(N(2, 3) = 8\), and \(N(2, r) = (r - 1)^2 + 2\) for \(r \geq 4\), then \(G = K_s \lor mK_p\) for some integers \(s \geq 0\), \(m \geq 2\), \(p \geq 1\) satisfying \(n = s + mp\) and \(r = s + 2(p - 1)\).
What is sufficiently large?

We know what “large” \((t, r)\)-regular graphs look like... but what the heck does large mean? Johnson and Morgan quantified “sufficiently large” for \((2, r)\)-regular graphs.

**Theorem ([3])**

Suppose that \(r \geq 1\) and \(G\) is a \((2, r)\)-regular graph on \(n\) vertices. If \(n \geq N(2, r)\), where \(N(2, 1) = 4\), \(N(2, 2) = 6\), \(N(2, 3) = 8\), and \(N(2, r) = (r - 1)^2 + 2\) for \(r \geq 4\), then \(G = K_s \vee mK_p\) for some integers \(s \geq 0\), \(m \geq 2\), \(p \geq 1\) satisfying \(n = s + mp\) and \(r = s + 2(p - 1)\).
What is sufficiently large?

Since \((t, r)\)-regular graphs for \(t > 2\) is not in the form \(K_s \lor mK_p\), we need the following notions.

\textbf{Definition (Shell and Kernel)}

The \(t\)-kernel of \(G\), \(\text{Ker}_t(G)\), is as the set of vertices that do not belong to any set of \(t\) independent vertices. The \(t\)-shell of \(G\), \(\text{Shell}_t(G) = V(G) \setminus \text{Ker}_t(G)\); that is, the \(t\)-shell is the set of vertices that are in some set of \(t\) independent vertices.

We use the \(t\)-kernel and \(t\)-shell to describe the form of large order \((t, r)\)-regular graphs, \(t > 2\). (the "almost" join of \(mK_p\) with a graph that is "almost" a clique).
What is sufficiently large?

Since \((t, r)\)-regular graphs for \(t > 2\) is not in the form \(K_s \lor mK_p\), we need the following notions.

**Definition (Shell and Kernel)**

- The \(t\)-kernel of \(G\), \(\text{Ker}_t(G)\), is as the set of vertices that do not belong to any set of \(t\) independent vertices.

- The \(t\)-shell of \(G\), \(\text{Shell}_t(G) = V(G) \setminus \text{Ker}_t(G)\); that is, the \(t\)-shell is the set of vertices that are in some set of \(t\) independent vertices.

We use the \(t\)-kernel and \(t\)-shell to describe the form of large order \((t, r)\)-regular graphs, \(t > 2\). (the “almost” join of \(mK_p\) with a graph that is “almost” a clique).
What is sufficiently large?

Since \((t, r)\)-regular graphs for \(t > 2\) is not in the form \(K_s \vee mK_p\), we need the following notions.

**Definition (Shell and Kernel)**

- The *t-kernel* of \(G\), \(\text{Ker}_t(G)\), is as the set of vertices that do not belong to any set of \(t\) independent vertices.
- The *t-shell* of \(G\), \(\text{Shell}_t(G) = V(G) \setminus \text{Ker}_t(G)\); that is, the *t-shell* is the set of vertices that are in some set of \(t\) independent vertices.
What is sufficiently large?

Since \((t, r)\)-regular graphs for \(t > 2\) is not in the form “\(K_s \lor mK_p\)”, we need the following notions.

Definition (Shell and Kernel)

- The \(t\)-kernel of \(G\), \(\text{Ker}_t(G)\), is as the set of vertices that do not belong to any set of \(t\) independent vertices.
- The \(t\)-shell of \(G\), \(\text{Shell}_t(G) = V(G) \setminus \text{Ker}_t(G)\); that is, the \(t\)-shell is the set of vertices that are in some set of \(t\) independent vertices.

We use the \(t\)-kernel and \(t\)-shell to describe the form of large order \((t, r)\)-regular graphs, \(t > 2\). (the “almost” join of \(mK_p\) with a graph that is “almost” a clique).
(3, 3)-regular graph; 
3-Ker(G) = \{0, 1, 2, 3, 4, 5, 6\}, 3 - Shell(G) = \{7, 8, 9\}. 
What is sufficiently large?

Jamison and Johnson gave the following bound.
What is sufficiently large?

Jamison and Johnson gave the following bound.

**Theorem ([2])**

Let $G$ be a $(t, r)$-regular graph with order $n$. Suppose that $t \geq 3$ and $r \geq 1$. For $n \geq N(t, r)$, $\langle \text{Shell}_t(G) \rangle \simeq mK_p$ for some integers $m \geq t$ and $p \geq 1$ such that $r = t(p - 1) + |\text{Ker}_t(G)|$. 
What is sufficiently large?

Jamison and Johnson gave the following bound.

**Theorem ([2])**

Let $G$ be a $(t, r)$-regular graph with order $n$. Suppose that $t \geq 3$ and $r \geq 1$. For $n \geq N(t, r)$, $\langle \text{Shell}_t(G) \rangle \simeq mK_p$ for some integers $m \geq t$ and $p \geq 1$ such that $r = t(p - 1) + |\text{Ker}_t(G)|$. The smallest $N(t, r) \leq \max[N(2, r) + r + t - 2, tr + 3r + t - 1]$.
What is sufficiently large?

Laffin and Özkan sharpened the bound for \((3, r)\)-regular graphs.
What is sufficiently large?

Laffin and Özkan sharpened the bound for \((3, r)\)-regular graphs.

**Theorem ([4])**

Suppose that \(r \geq 1\), \(G\) is a \((3, r)\)-regular graph of order \(n\). Suppose that \(n \geq N(3, r)\) where \(N(3, 1) = 5\), \(N(3, 2) = 7\), \(N(3, 3) = 9\), \(N(3, 4) = 16\) and \(N(3, r) = (r - 1)^2 + r + 2\) for \(r \geq 5\). Then \(\langle \text{Shell}_3(G) \rangle \cong mK_p\) for some integers \(m \geq 3\) and \(p \geq 1\) such that \(r = 3(p - 1) + |\text{Ker}_3(G)|\).

From now on \(\text{Ker}(G) = \text{Ker}_3(G)\) and \(\text{Shell}(G) = \text{Shell}_3(G)\).
(3, r)-regular graphs

We will prove the bound given in the previous slide for \( r > 4 \).
We will prove the bound given in the previous slide for $r > 4$. First, some important observations.
We will prove the bound given in the previous slide for \( r > 4 \). First, some important observations.

**Proposition (1)**

If \( G \) is a \((t, r)\)-regular graph, then \( |\text{Ker}(G)| \leq r \).

**Proposition (2)**

If \( G \) is a \((t, r)\)-regular graph, and \( u \in \text{Shell}(G) \) then \( |N(u)| \leq r \).

**Proposition (3)**

Let \( G \) be a \((3, r)\)-regular graph on \( n \geq r^2 + r + 3 \) vertices. Let \( u \in \text{Shell}(G) \), and \( |N(u)| = k \). Then \( G \setminus N[u] \) is a \((2, r-k)\)-regular graph of the form \( K_s \lor mK_p \) where \( s + 2(p-1) = r-k \).
We will prove the bound given in the previous slide for $r > 4$. First, some important observations.

**Proposition (1)**

*If $G$ is a $(t, r)$-regular graph, then $|\text{Ker}(G)| \leq r$.**

**Proposition (2)**

*If $G$ is a $(t, r)$-regular graph, and $u \in \text{Shell}(G)$ then $|N(u)| \leq r$.***
We will prove the bound given in the previous slide for $r > 4$. First, some important observations.

**Proposition (1)**

*If* $G$ *is a* $(t, r)$-*regular graph, then* $|\text{Ker}(G)| \leq r$.

**Proposition (2)**

*If* $G$ *is a* $(t, r)$-*regular graph, and* $u \in \text{Shell}(G)$ *then* $|N(u)| \leq r$.

**Proposition (3)**

*Let* $G$ *be a* $(3, r)$-*regular graph on* $n \geq r^2 + r + 3$ *vertices. Let* $u \in \text{Shell}(G)$, *and* $|N(u)| = k$. *Then* $G \setminus N[u]$ *is a* $(2, r - k)$-*regular graph of the form* $K_s \lor mK_p$ *where* $s + 2(p - 1) = r - k$. 

(3, r)-Regular Graphs

Proof.

- Consider $\{u, v, w\} \in Shell(G)$, $vw, uw \in E(G)$. 

(3, r)-Regular Graphs

Proof.

- Consider \( \{u, v, w\} \in Shell(G), vw, uw \in E(G) \).

- We observe that \( |N[u, v, w]| < |Shell(G)| \), and let \( z \in Shell(G) \setminus N[u, v, w] \) have degree \( k \).
Proof.

- Consider \( \{u, v, w\} \in Shell(G) \), \( vw, uw \in E(G) \).

- We observe that \( |N[u, v, w]| < |Shell(G)| \), and let \( z \in Shell(G) \setminus N[u, v, w] \) have degree \( k \).
Proof continued.

- $G \setminus N[z]$ must be a $(2, r - k)$ graph of the form $K_s \lor mK_p$ (by Proposition (3)).

Calculate the degree of $w$ to see $d(w) > r$ implying $w \not\in \text{Shell}(G)$. Contradiction.

Then $uv$ must be an edge and $\text{Shell}(G)$ must be a disjoint union of cliques $mK_p$.

If $X$ is a 3 vertex independent set, the vertices need to be adjacent to $3(p - 1)$ vertices in $\text{Shell}(G)$ and adjacent to every element of $\text{Ker}(G)$.
(3, r)-Regular Graphs

Proof continued.

- $G \setminus N[z]$ must be a $(2, r - k)$ graph of the form $K_s \vee mK_p$ (by Proposition (3)).
- It must be the case that $w \in K_s$ and $u, v$ in different $K_p$.

Calculate the degree of $w$ to see $d(w) > r$ implying $w \not\in \text{Shell}(G)$. Contradiction.

Then $uv$ must be an edge and $\text{Shell}(G)$ must be a disjoint union of cliques $mK_p$.

If $X$ is a 3 vertex independent set, the vertices need to be adjacent to $3(p - 1)$ vertices in $\text{Shell}(G)$ and adjacent to every element of $\text{Ker}(G)$. 

Proof continued.

- $G \backslash N[z]$ must be a $(2, r - k)$ graph of the form $K_s \vee mK_p$ (by Proposition (3)).

- It must be the case that $w \in K_s$ and $u, v$ in different $K_p$.

- Calculate the degree of $w$ to see $d(w) > r$ implying $w \notin Shell(G)$. Contradiction.

\begin{center}
\begin{tikzpicture}
  \node (w) at (0,0) [circle, draw] {$w$};
  \node (u) at (-1,-1) [circle, draw] {$u$};
  \node (v) at (1,-1) [circle, draw] {$v$};
  \node (vkp) at (0,-2) [circle, draw] {$\cdots$};
  \node (ks) at (0,2) [circle, draw] {$K_s$};
  \node (mkg) at (0,-2) [circle, draw] {$mK_p$};

  \draw (w) -- (u);
  \draw (w) -- (v);
  \draw (w) -- (vkp);
  \draw (ks) -- (mkg);
\end{tikzpicture}
\end{center}
Proof continued.

- $G \setminus N[z]$ must be a $(2, r - k)$ graph of the form $K_s \vee mK_p$ (by Proposition (3)).
- It must be the case that $w \in K_s$ and $u, v$ in different $K_p$.

\[ K_s \quad \quad \quad \quad \quad \quad w \quad \quad \quad \quad \quad \quad u \quad v \quad \ldots \]

- Calculate the degree of $w$ to see $d(w) > r$ implying $w \notin Shell(G)$. Contradiction.
- Then $uv$ must be an edge and $Shell(G)$ must be a disjoint union of cliques $mK_p$. 
Proof continued.

- $G \setminus N[z]$ must be a $(2, r - k)$ graph of the form $K_s \lor mK_p$ (by Proposition (3)).
- It must be the case that $w \in K_s$ and $u, v$ in different $K_p$.

\[ \text{K}_s \quad \text{w} \]
\[ m\text{K}_p \quad \text{u} \quad \text{v} \quad \ldots \]

- Calculate the degree of $w$ to see $d(w) > r$ implying $w \not\in \text{Shell}(G)$. Contradiction.
- Then $uv$ must be an edge and $\text{Shell}(G)$ must be a disjoint union of cliques $mK_p$.
- If $X$ is a 3 vertex independent set, the vertices need to be adjacent to $3(p - 1)$ vertices in $\text{Shell}(G)$ and adjacent to every element of $\text{Ker}(G)$. 
Theorem

A graph is (3, 1)-regular if and only if $G = K_2 \cup \overline{K}_2$, $G = K_1 \cup P_2$, and $G = \alpha K_1 \cup K_{1,m}$ where $m \geq 3$ and $\alpha = \{0, 1\}$.
Theorem

A graph is \((3, 1)\)-regular if and only if \(G = K_2 \cup \overline{K}_2\), \(G = K_1 \cup P_2\), and \(G = \alpha K_1 \cup K_{1,m}\) where \(m \geq 3\) and \(\alpha = \{0, 1\}\).

By Jamison-Johnson \(N(3, 1) \leq 8\).
(3, 1)-Regular Graphs

**Theorem**

A graph is (3, 1)-regular if and only if $G = K_2 \cup \overline{K}_2$, $G = K_1 \cup P_2$, and $G = \alpha K_1 \cup K_{1,m}$ where $m \geq 3$ and $\alpha = \{0, 1\}$.

By Jamison-Johnson $N(3, 1) \leq 8$.

**Corollary**

The only (3, 1)-regular graph with $\langle \text{Shell}(G) \rangle \cong mK_p$ is $K_2 \cup \overline{K}_2$. Thus $N(3, 1) = 5$. 
(3, 2)-Regular Graphs

By Jamison-Johnson $N(3, 2) \leq 14$. 

Examples: 

M.R. Laffin (Michigan Tech)
(3, 2)-Regular Graphs

By Jamison-Johnson \( N(3, 2) \leq 14 \).

**Theorem**

*If \( G \) is a \((3, 2)\)-regular graph on \( n \geq 7 \) vertices then \( \langle \text{Shell}(G) \rangle \cong mK_1 \) for some integer \( m \geq 3 \) and \( |\text{Ker}(G)| = 2 \).*
(3, 2)-Regular Graphs

By Jamison-Johnson $N(3, 2) \leq 14$.

**Theorem**

*If $G$ is a $(3, 2)$-regular graph on $n \geq 7$ vertices then $\langle \text{Shell}(G) \rangle \cong mK_1$ for some integer $m \geq 3$ and $|\text{Ker}(G)| = 2$.***

Examples:
(3, 2)-Regular Graphs

**Theorem**

If $G$ is $(3, 2)$-regular on 5 or 6 vertices, the $G$ must be one of the following graphs: $K_{2,3}, K_{2,4}, K_{2,3} - e, K_{2,4} - e$ where $e$ is an edge, $K_2 \lor \overline{K}_3, (K_2 \lor \overline{K}_3) \cup K_1, K_2 \lor \overline{K}_4, (H \lor \overline{K}_3) - e^*, (H \lor \overline{K}_4) - e^*$ where $e^* \not\subseteq E(H)$ and $H = K_2, C_4 \cup K_1, C_4 \cup \overline{K}_2, P_4, P_3 \cup K_1, P_2 \cup K_2, (K_4 - e) \cup K_1, K_3 \cup \overline{K}_2, 2K_2 \cup K_1$ or:
By Jamison-Johnson, $N(3, 3) \leq 20$. 
By Jamison-Johnson, $N(3, 3) \leq 20$.

**Theorem**

If $G$ is $(3, 3)$-regular with order $n \geq 9$, then $\langle \text{Shell}(G) \rangle \cong mK_p$ for $m \geq 3$ and $p \in \{1, 2\}$.

Some sporadic $(3, 3)$ graphs include $K_2 \cup P_3$, $K_1 \cup K_2$, $C_5 \cup K_1$. 
Computer Search for \((3, 3)\)-Regular Graphs

We performed a (naive) computer search to find all \((3, 3)\)-regular graphs on 6 to 10 vertices.

<table>
<thead>
<tr>
<th>Number of Vertices</th>
<th>Sage</th>
<th>nauty</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>20 minutes</td>
<td>–</td>
</tr>
<tr>
<td>7</td>
<td>1 hour</td>
<td>–</td>
</tr>
<tr>
<td>8</td>
<td>6 hours</td>
<td>–</td>
</tr>
<tr>
<td>9</td>
<td>18 hours</td>
<td>.2 seconds</td>
</tr>
<tr>
<td>10</td>
<td>–</td>
<td>1 second</td>
</tr>
</tbody>
</table>
Future Work & Open Problems

1. Sharpen any of the preexisting bounds! (Morgan-Johnson, Jamison-Johnson, Laffin-Özkan)

2. Characterize all small order \((t,r)\)-regular graphs for varying values of \(t\) and \(r\)

3. Study \((3,r)\)-regular graphs where any 3 vertices have exactly \(r\) neighbors

4. Study possible relationships between strongly regular graphs and \((t,r)\)-regular graphs
Future Work & Open Problems

1. Sharpen any of the preexisting bounds! (Morgan-Johnson, Jamison-Johnson, Laffin-Özkan)

2. Characterize all small order \((t, r)\)-regular graphs for varying values of \(t\) and \(r\)

3. Study \((3, r)\)-regular graphs where any 3 vertices have exactly \(r\) neighbors

4. Study possible relationships between strongly regular graphs and \((t, r)\)-regular graphs
Future Work & Open Problems

1. Sharpen any of the preexisting bounds! (Morgan-Johnson, Jamison-Johnson, Laffin-Özkan)

2. Characterize all small order \((t, r)\)-regular graphs for varying values of \(t\) and \(r\)

3. Study \((3, r)\)-regular graphs where any 3 vertices have exactly \(r\) neighbors
Future Work & Open Problems

1. Sharpen any of the preexisting bounds! (Morgan-Johnson, Jamison-Johnson, Laffin-Özkan)

2. Characterize all small order \((t, r)\)-regular graphs for varying values of \(t\) and \(r\)

3. Study \((3, r)\)-regular graphs where any 3 vertices have exactly \(r\) neighbors

4. Study possible relationships between strongly regular graphs and \((t, r)\)-regular graphs

