Independent sets in graphs with given minimum degree

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An extremal question for independent sets

**Independent set**: Set of pairwise non-adjacent vertices

\[ i(G) \]: Number of independent sets in \( G \)

**Question**

*Fix a family \( \mathcal{G} \) of graphs. What is the maximum (or minimum) of \( i(G) \) as \( G \) ranges over \( \mathcal{G} \)?
Some previous results

Theorem (Prodinger-Tichy, Cutler-Radcliffe, Kahn, Zhao, Hua)

1. $\mathcal{G} = \{\text{trees on a fixed number of vertices}\}$
   - $i(G)$ maximized by star and minimized by path

2. $\mathcal{G} = \{\text{graphs on fixed number of vertices and edges}\}$
   - $i(G)$ maximized by lex graph

3. $\mathcal{G} = \{\text{n-vertex, d-regular graphs}\}$
   - $i(G)$ maximized by union of $n/2d$ copies of $K_{d,d}$

4. $\mathcal{G} = \{\text{n-vertex graphs (n ≥ 6) without cut-edge}\}$
   - $i(G)$ maximized by $K_{2,n-2}$

Also forests, unicyclic graphs, graphs with a given number of cut-edges, ...
Today’s family of interest

\[ \mathcal{G}_n(\delta) = \{n\text{-vertex graphs with minimum degree at least } \delta\} \]

**Speculation:** removing edges increases independent set count, so

\[ i(G) \text{ maximized in } \mathcal{G}_n(\delta) \text{ by union of } K_{\delta,\delta}'s \]

**Not true:** E.g., \( \delta = 1 \)

- \( G_{\text{match}} \) a disjoint union of \( n/2 \) edges \( \rightarrow i(G_{\text{match}}) = 3^{\frac{n}{2}} \)
- \( G_{\text{star}} \) a star on \( n \) vertices \( \rightarrow i(G_{\text{star}}) = 2^{n-1} + 1 \)
- For all large \( n \) \( \rightarrow i(G_{\text{star}}) > i(G_{\text{match}}) \)
An unbalanced maximizer

**Theorem (G., 2010)**

For \( n \geq 16\delta^2 \) and \( G \in \mathcal{G}_n(\delta) \),

\[
i(G) \leq i(K_{\delta,n-\delta}) = 2^{n-\delta} + 2^\delta - 1
\]

with equality iff \( G = K_{\delta,n-\delta} \)

**Remark**: related to some previous results

trees \( \subseteq \mathcal{G}_n(1) \)

graphs without a cut edge \( \subseteq \mathcal{G}_n(2) \)

So we extend results of Prodinger-Tichy, Hua
Proof of theorem — three regimes

For $G \in \mathcal{G}_n(\delta)$:

\[
\begin{cases}
I \text{ an independent set of maximum size} \\
J \text{ the complement of } I
\end{cases}
\]

Three regimes:

1. $|J| \leq \delta$
   - Easy regime

2. $\delta < |J| < 4\delta$
   - Where most of the work happens

3. $|J| \geq 4\delta$
   - Non-trivial, but easy given some previous results
Regime 1 — when $|J| \leq \delta$

- Must have $|J| = \delta$
  - If $|J| < \delta$, vertices of $I$ violate min degree condition

- $i(G) = 2^{n-\delta} + i(G[J]) - 1$

- Maximized uniquely when
  - $G[J]$ has no edges
  - $G = K_{\delta, n-\delta}$
Regime 3 — when $|J| \geq 4\delta$

Theorem (Alekseev)

$G$ an $n$-vertex graph with all independent sets having size at most $\alpha$

$$i(G) \leq \left(1 + \frac{n}{\alpha}\right)^{\alpha}$$

When $|J| \geq 4\delta$ have $\alpha \leq n - 4\delta$ so

$$i(G) \leq \left(1 + \frac{n}{n - 4\delta}\right)^{n - 4\delta} \leq 2^{n - \delta} \left(\frac{e^2}{8}\right)^{\delta} < 2^{n - \delta} < i(K_{\delta, n - \delta})$$
Regime 3 — when \( \delta < |J| < 4\delta \)

Split \( J \) into two sets

\[
J' = \text{the } \delta \text{ vertices in } J \text{ with largest degrees to } I
\]

\[
J \setminus J' = \text{the rest of } J
\]

By minimum degree condition:

- \(#\) (edges from \( I \) to \( J \)) \( \geq \) \((n - |J|)\delta\)

Easy (reverse) Markov bound:

- All vertices in \( J' \) have at least \( \frac{n - |J|}{|J| - \delta + 1} \) neighbours in \( I \)
Regime 3 — independent sets disjoint from $J'$

\[
\#(\text{independent sets disjoint from } J') = \sum_{S \subseteq J \setminus J' : G[S] \text{ empty}} 2^{n - |J| - |N(S)|}
\]

Since $I$ is maximal, have $|N(S)| \geq |S|$

\[
\#(\text{independent sets disjoint from } J') \leq 2^{n - |J|} \left( \frac{3}{2} \right)^{|J \setminus J'|}
\]

\[
= 2^{n - \delta} \left( \frac{3}{4} \right)^{|J|}
\]

\[
\leq \left( \frac{3}{4} \right) 2^{n - \delta}
\]
Regime 3 — independent sets meeting $J'$

$$\#(\text{independent sets meeting } J') = \sum_{\emptyset \neq S \subseteq J' : G[S] \text{ empty}} 2^{n-|J'|-|N(S)|}$$

By definition of $J'$, have $|N(S)| \geq \frac{n-|J|}{|J|-\delta + 1}$

$$\#(\text{independent sets meeting } J') \leq 2^{n-\delta} 2^\delta - \frac{n-|J|}{|J|-\delta + 1}$$

$$\leq 2^{n-\delta} 2^\delta - \frac{n-4\delta}{3\delta + 1}$$

$$\leq \left( \frac{1}{4} \right) 2^{n-\delta}$$

(the last as long as $n \geq 16\delta^2$)
Summary

Regime 1: $|J| \leq \delta$
- $i(G) < i(K_{\delta,n-\delta})$ unless $G = K_{\delta,n-\delta}$
- Valid for all $n$ and $\delta$

Regime 2: $|J| \geq 4\delta$
- $i(G) \leq 2^{n-\delta} < i(K_{\delta,n-\delta})$
- Valid as long as $n \geq 4\delta$

Regime 3: $\delta < |J| < 4\delta$
- $i(G) \leq \left(\frac{3}{4}\right) 2^{n-\delta} + \left(\frac{1}{4}\right) 2^{n-\delta} < i(K_{\delta,n-\delta})$
- Valid as long as $n \geq 16\delta^2$
A slightly stronger result

**Theorem (G., 2010)**

For $n \geq 2.6\delta^2 + 9.8\delta + 2$ and $G \in \mathcal{G}_n(\delta)$,

$$i(G) \leq i(K_{\delta,n-\delta})$$

with equality iff $G = K_{\delta,n-\delta}$

**Stable set polynomial:** $P(G, x) = \sum_{I} an\ independent\ set \; x^{|I|}$

**Theorem (G., 2010)**

For $n \geq f(x, \delta)$ and $G \in \mathcal{G}_n(\delta)$,

$$P(G, x) \leq P(K_{\delta,n-\delta}, x)$$

with equality iff $G = K_{\delta,n-\delta}$
Some conjectures/questions

A conjecture

- $K_{\delta, n-\delta}$ remains maximizer all the way to $n \geq 2\delta$

Some questions

- What happens when $\delta > n/2$?
- Which graph in $G_n(\delta)$ maximizes number of independent sets of a particular fixed size $t$?

  - $\delta = 1$: $K_{1, n-1}$ is maximizer for all $t \geq 3$, but not for $t = 2$
  - $\delta = 2$: Small case analysis suggests $K_{2, n-2}$ is maximizer for all $t \geq 4$, but not for $t = 2, 3$
  - Wild speculation: For general $\delta$, $K_{\delta, n-\delta}$ is maximizer for all $t \geq \delta + 2$