Revolutionaries and Spies II:
Hypercubes & Complete Multipartite Graphs

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slides available on DBW preprint page

Joint work with
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A Game of National Security

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**Ques.** Fix $G, m, r$. How many spies are needed to win?
Spy-Good Graphs

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- Chordal graphs?
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**Thought:** spy-bad means dense enough and sparse enough for revs to threaten some unreachable mtg.
Random Graphs

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The revs meet at $v$ in the first move and win.
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Hypercubes

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$\leq r - 5$ spies at singles leave too many threats at doubles (spies at triples reach only three doubles).
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Revs start at $r$ singles, threatening at $\binom{r}{2}$ doubles. $r - 2$ spies at singles can’t reach all threats at doubles.

$r - 4$ spies at singles leave six threats at doubles, not reached by two triples (two triangles don’t cover $E(K_4)$).
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$\therefore r - 3$ spies occupy singles, plus one at a triple. By symmetry, spy is at 123, with the others at 4, ..., $r$. 
Revs move to win

Revs at 1 and 2 move to $\emptyset$.
For $3 \leq j \leq r$, the rev at $j$ moves to $jd$. 
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But, revs at $3d$ and $jd$ threaten $3jd$ on next move, and no other spy can reach a neighbor of $3jd$ now.
Smaller dimensions

When $d > r$, revs beat $r - 2$ spies on $Q_d$ when $m = 2$. On smaller hypercubes, revs do almost as well.
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**Thm.** If $(d - 1)2^{\lfloor d/11 \rfloor} \geq r$, then $r$ revs beat $r - \left\lceil \frac{r}{d-1} \right\rceil - 1$ spies on $Q_d$ when $m = 2$. 
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Allocate \( r_i \) revolutionaries to each \( x_i \in X \), where \( r_i < d \). Using \( x_i \) as \( \emptyset \), they play the earlier strategy around \( x_i \).
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∴ revs win against fewer than $r - t$ spies.

Since $(d - 1)t \geq r$, the revs win if $s < r - \frac{r}{d-1}$. 

\[ \square \]
Complete $k$-partite graphs

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Spies win on $G_k$ if $s \geq \frac{k}{k-1} \frac{r}{m} + k$. 
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How many spies are needed to avoid losing by swarm?
Case 1: $s_i > t$ for some $i$; revs swarm to part $i$. New meetings use $m$ incoming revs., not guardable by spies from part $i$. At least $\lfloor (k-1)t/m \rfloor$ additional spies must come from other parts, so

$$s \geq s_i + \left\lfloor \frac{(k-1)t}{m} \right\rfloor \geq t \left[ 1 + \frac{k-1}{m} \right] = \frac{k-1+m}{k} \frac{r}{m}.$$
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Part $i$ has $t - s_i$ partial meetings; $i$-swarm can fill them (since $s_i \geq 0$) if $(k-1)t \geq t(m-1)$, implied by $k \geq m$. 
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Hence spies from other parts must guard $\lfloor (r - s_i)/m \rfloor$ new meetings. Summing $s - s_i \geq \frac{r-s_i-m+1}{m}$ yields

$$(k-1+\frac{1}{m})s > k \frac{r-m+1}{m}, \text{ so } s > \frac{k(r-m+1)}{m(k-1)+1} > \frac{k}{k-1} \frac{r}{m+c} - k.$$
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Hence spies from other parts must guard $\left\lfloor (r - s_i)/m \right\rfloor$ new meetings. Summing $s - s_i \geq \frac{r - s_i - m + 1}{m}$ yields

$$(k-1+\frac{1}{m})s > kr - m + 1, \text{ so } s > \frac{k(r-m+1)}{m(k-1)+1} > \frac{k}{k-1} \frac{r}{m+c} - k.$$

When $k \geq m$, the requirement from Case 2 is weaker (better for spies) than from Case 1.
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**Def.** The $m$ revs in an $m$-meeting and one spy on them are **bound**; others are **free**. Currently in part $i$, let $r_i = \#\text{free revs}, s_i = \#\text{free spies}$. Also $\hat{r} = \text{total } \#\text{free revs}, \hat{s} = \text{total } \#\text{free spies}$. 
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**Pf.** Hall’s Theorem yields a matching that covers new \( m \)-meetings with free spies who can move there.
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** Conj.** For fixed $m$, the threshold for the number of spies needed to win is asymptotic to $1.5 \frac{r}{m}$. 