The Gold Grabbing Game

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Joint Work with Tyler Seacrest

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Rules of the Game

- A tree has (integer) amounts of gold at each vertex
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- Alice moves when there is an even number of vertices left and Bob moves when there is an odd number.
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• A tree has (integer) amounts of gold at each vertex (and an even number of vertices).
• Alice moves when there is an even number of vertices left and Bob moves when there is an odd number.
• On each turn, a player removes a leaf and adds the associated amount of gold to his or her score.
Advantages of the Game

- First developed (on a path) at 1996 International Olympiad in Informatics
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- David Ginat used this as a game (on a path) to teach college mathematics and computer science
- Can be used at a variety of levels and with “mis-matched” opponents
Greedy Gold Grabbing on a Path

Alice: Bob:
Greedy Gold Grabbing on a Path

Alice: 5
Bob: 5

4 8 7 1 9

[Diagram of a path with numbers 4, 8, 7, 1, 9, and 5, illustrating the game of Greedy Gold Grabbing]
Greedy Gold Grabbing on a Path

Alice: 5
Bob: 9
Greedy Gold Grabbing on a Path

Alice: 5 + 4  Bob: 9
Greedy Gold Grabbing on a Path

Alice: 5 + 4
Bob: 9 + 8
Greedy Gold Grabbing on a Path

Alice: $5 + 4 + 7$  Bob: $9 + 8$
Greedy Gold Grabbing on a Path

Alice: 5 + 4 + 7  Bob: 9 + 8 + 1
Greedy Gold Grabbing on a Path

Alice: 16
Bob: 18
Greedy Gold Grabbing on a Path Can Backfire

Alice: 4 8 7 1 99 5
Bob: 4 8 7 1 99 5

Alice: Bob:
Greedy Gold Grabbing on a Path Can Backfire

A diagram is shown of a sequence of numbers connected by dots, with Alice choosing the number 4 and Bob choosing the number 5.

Referencing the text, the diagram illustrates a path where Alice and Bob select numbers. The sequence is: 4, 8, 7, 1, 99, 5. Alice's choice is marked with a grey dot, and Bob's is a black dot. Alice grabs 4, and Bob grabs 5.
Greedy Gold Grabbing on a Path Can Backfire

Alice: 4  Bob: 8
Greedy Gold Grabbing on a Path Can Backfire

Alice: 4 + 7  Bob: 8
Greedy Gold Grabbing on a Path Can Backfire
Greedy Gold Grabbing on a Path Can Backfire

Alice: $4 + 7 + 99$  
Bob: $8 + 1$
Greedy Gold Grabbing on a Path Can Backfire

Alice: 4 + 7 + 99  Bob: 8 + 1 + 5
Greedy Gold Grabbing on a Path Can Backfire

Alice: 110  Bob: 14
Clever Gold Grabbing on a Path

Alice:  

Bob:  

4 8 7 1 9 5

Rules

Gold Grabbing on Paths

Gold Grabbing on Trees

Lemma

Theorem

References
Clever Gold Grabbing on a Path

Alice: Bob:

4 8 7 1 9 5

References
Clever Gold Grabbing on a Path

Alice: 4  
Bob:
Clever Gold Grabbing on a Path

Alice: 4  Bob: 8
Clever Gold Grabbing on a Path

Alice: 4 + 7
Bob: 8
Clever Gold Grabbing on a Path

Alice: $4 + 7$  
Bob: $8 + 5$
Clever Gold Grabbing on a Path

Alice: $4 + 7 + 9$  Bob: $8 + 5$
Clever Gold Grabbing on a Path

Alice: $4 + 7 + 9$    Bob: $8 + 5 + 1$
Clever Gold Grabbing on a Path

Alice: 20  
Bob: 14
Why $n$ Must Be Even

Alice wins

Bob wins
Extending to a Tree
Extending to a Tree

Alice: 8  Bob: 8
Extending to a Tree

Alice: 8  Bob: 7

9 2 5 4 6

4 1 3 7

4

References
Extending to a Tree

Alice: 14  Bob: 7
Extending to a Tree

Alice: 14
Bob: 11
Extending to a Tree

Alice: 19  Bob: 11
Extending to a Tree
Extending to a Tree

Alice: 28  Bob: 15
Extending to a Tree

Alice: 28  Bob: 18
Extending to a Tree

Alice: 30  Bob: 18
Extending to a Tree

Alice: 30  Bob: 19
Past Work

Theorem (Micek and Walczak, 2010)

*If* $T$ *is a tree with an even number of vertices, Alice can secure at least* **one-fourth** *of the gold.*
Past Work

Theorem (Micek and Walczak, 2010)

If $T$ is a tree with an even number of vertices, Alice can secure at least one-fourth of the gold.

Conjecture (Micek and Walczak, 2010)

If $T$ is a tree with an even number of vertices, Alice can secure at least half the gold.
Theorem

If $T$ is a tree with an even number of vertices, Alice can secure at least half the gold.
Rooted and Non-rooted Gold Grabbing on a Tree

Alice: 4 9 2 5 4 8
Bob: 1 7 3

References

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Rooted and Non-rooted Gold Grabbing on a Tree

Alice: 4
       9 2 5 4
       8
       6
       7 3

Bob:  1
      3
      4
      6
      8

References
Rooted and Non-rooted Gold Grabbing on a Tree

Alice: 7 + 1   Bob: 3 + 2
Rooted and Non-rooted Gold Grabbing on a Tree

Alice: 7 3
Bob: 1

Alice: 2
Bob: 1 3

Rooted and Non-rooted Gold Grabbing on a Tree

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Rules
Gold Grabbing on Paths
Gold Grabbing on Trees
Lemma
Theorem
References
Rooted and Non-rooted Gold Grabbing on a Tree

Alice: $7 + 2$  
Bob: $3 + 1$
Lemma

For Alice, the non-rooted game on any number of vertices is at least as good as the rooted game.
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Proof.

- Induct on the number of vertices.
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- Induct on the number of vertices.
- True for $n \leq 3$. 
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Proof.

- Induct on the number of vertices.
- True for $n \leq 3$.
- True when Alice moves first.
Lemma

For Alice, the non-rooted game on any number of vertices is at least as good as the rooted game.

Proof.

- Induct on the number of vertices.
- True for $n \leq 3$.
- True when Alice moves first.
- True when Bob’s optimal first move (in the non-rooted game) isn’t the root.
Lemma

For Alice, the non-rooted game on any number of vertices is at least as good as the rooted game.

Proof.

- Induct on the number of vertices.
- True for \( n \leq 3 \).
- True when Alice moves first.
- True when Bob’s optimal first move (in the non-rooted game) isn’t the root.
- True when the root is adjacent to vertex \( u \) of degree 2.
Lemma

For Alice, the non-rooted game on any number of vertices is at least as good as the rooted game.

Proof.

- Induct on the number of vertices.
- True for $n \leq 3$.
- True when Alice moves first.
- True when Bob’s optimal first move (in the non-rooted game) isn’t the root.
- True when the root is adjacent to vertex $u$ of degree 2.
- True otherwise.
Theorem

If $T$ has an even number of vertices, Alice can secure at least half the gold.
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Proof.

- Let $v$ be any leaf, $u$ the neighbor of $v$, and $T' = T - v$. 
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Proof.

- Let $v$ be any leaf, $u$ the neighbor of $v$, and $T' = T - v$.
- **Game 1**: Alice goes first on $T'$ rooted at $u$. Bob gets $v$.
Theorem

If $T$ has an even number of vertices, Alice can secure at least half the gold.

Proof.

- Let $v$ be any leaf, $u$ the neighbor of $v$, and $T' = T - v$.
- Game 1: Alice goes first on $T'$ rooted at $u$. Bob gets $v$.
- Game 2: Alice gets $v$ and goes second on $T'$ rooted at $u$. 
Theorem

*If* $T$ *has an even number of vertices, Alice can secure at least half the gold.*

**Proof.**

- Let $v$ be any leaf, $u$ the neighbor of $v$, and $T' = T - v$.
- **Game 1:** Alice goes first on $T'$ rooted at $u$. Bob gets $v$.
- **Game 2:** Alice gets $v$ and goes second on $T'$ rooted at $u$.
  - By choosing between the games, Alice can win or tie.
Theorem

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  - By choosing between the games, Alice can win or tie.
- Game 1: equivalent to playing on $T$ rooted at $v$. 
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- **Game 1**: Alice goes first on $T'$ rooted at $u$. Bob gets $v$.
- **Game 2**: Alice gets $v$ and goes second on $T'$ rooted at $u$.
  - By choosing between the games, Alice can win or tie.
- **Game 1**: equivalent to playing on $T$ rooted at $v$. By lemma, playing non-rooted is at least as good.
If $T$ has an even number of vertices, Alice can secure at least half the gold.

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- **Game 1**: Alice goes first on $T'$ rooted at $u$. Bob gets $v$.
- **Game 2**: Alice gets $v$ and goes second on $T'$ rooted at $u$.
  - By choosing between the games, Alice can win or tie.
- Game 1: equivalent to playing on $T$ rooted at $v$. By lemma, playing non-rooted is at least as good.
- Game 2: equivalent to her taking $v$ and then playing on $T'$ rooted at $u$. 

Theorem

If $T$ has an even number of vertices, Alice can secure at least half the gold.

Proof.

- Let $v$ be any leaf, $u$ the neighbor of $v$, and $T' = T - v$.
- **Game 1**: Alice goes first on $T'$ rooted at $u$. Bob gets $v$.
- **Game 2**: Alice gets $v$ and goes second on $T'$ rooted at $u$.
  - By choosing between the games, Alice can win or tie.
- Game 1: equivalent to playing on $T$ rooted at $v$. By lemma, playing non-rooted is at least as good.
- Game 2: equivalent to her taking $v$ and then playing on $T'$ rooted at $u$. By lemma, playing non-rooted is at least as good.
Questions and Extensions

- What is Alice’s strategy?
Questions and Extensions

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- What if a tree (or even a path) has an odd number of vertices?
References


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