May 14, 2011

24th Cumberland Conference

Planarity for Partially Ordered Sets

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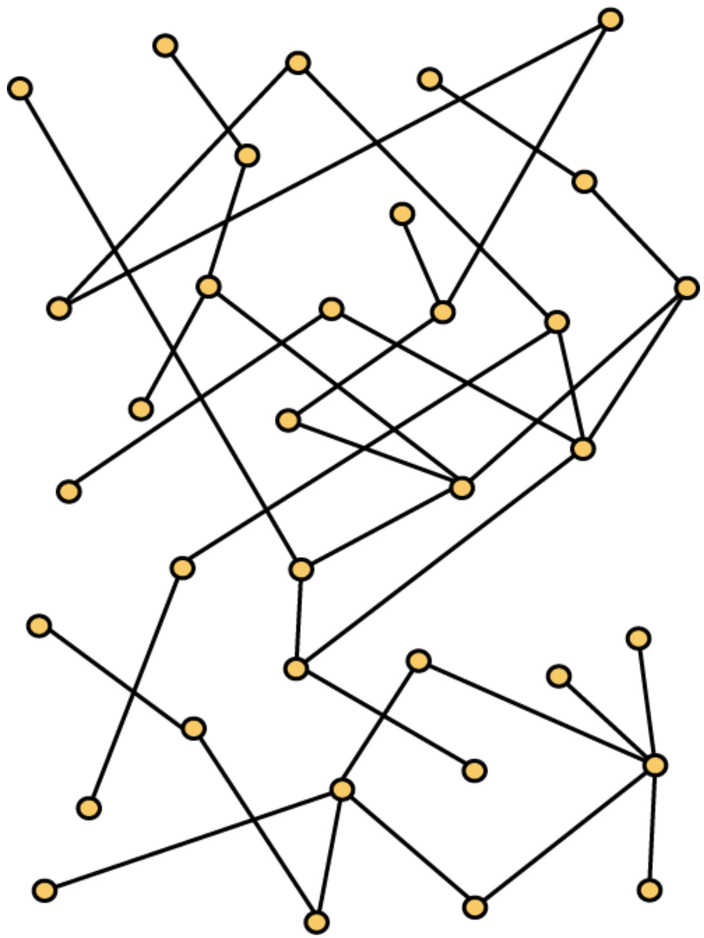
Challenge Problem – For a Glass of Wine



Problem Find the dimension of this poset.

Solutions by Email. Three winners.

Competition limited to grad students, postdocs and assistant professors.

Planarity for Graphs – Well Understood

Theorem (Kuratowski) A graph G is planar if and only if it does not contain a homeomorph of K_5 or $K_{3,3}$.

Fact Given a graph G, the question "Is G planar?" can be answered with an algorithm whose running time is linear in the number of edges in G.

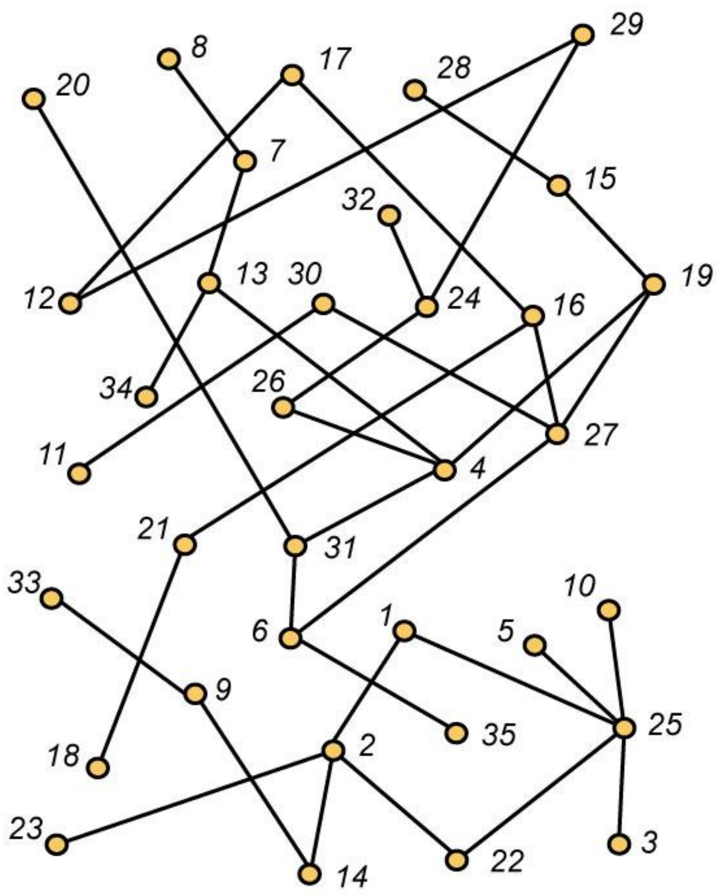
Partially Ordered Sets - Posets

Definition A poset P consists of a ground set and a binary relation $\leq$ which is reflexive, antisymmetric and transitive.

Example As ground set, take any family of sets, and set $A \leq B$ if and only if A is a subset of B.

Example As ground set, take any set of positive integers and set $m \leq n$ if and only if m divides n without remainder.

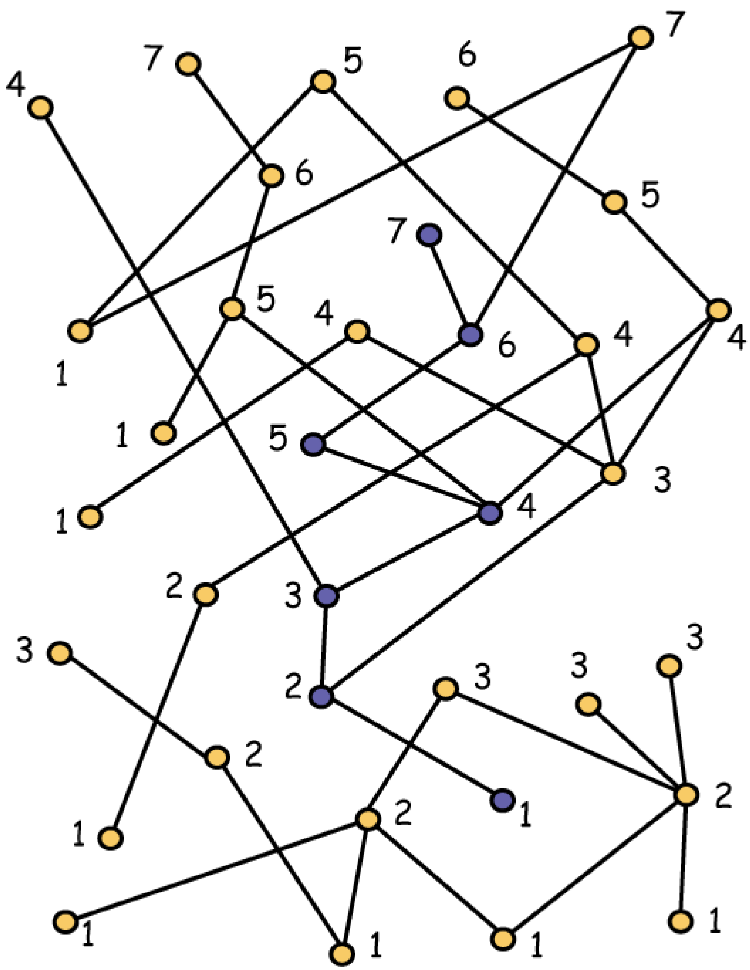
Order Diagrams for Posets



In this poset, 18 < 17 while 33 is incomparable to 19.

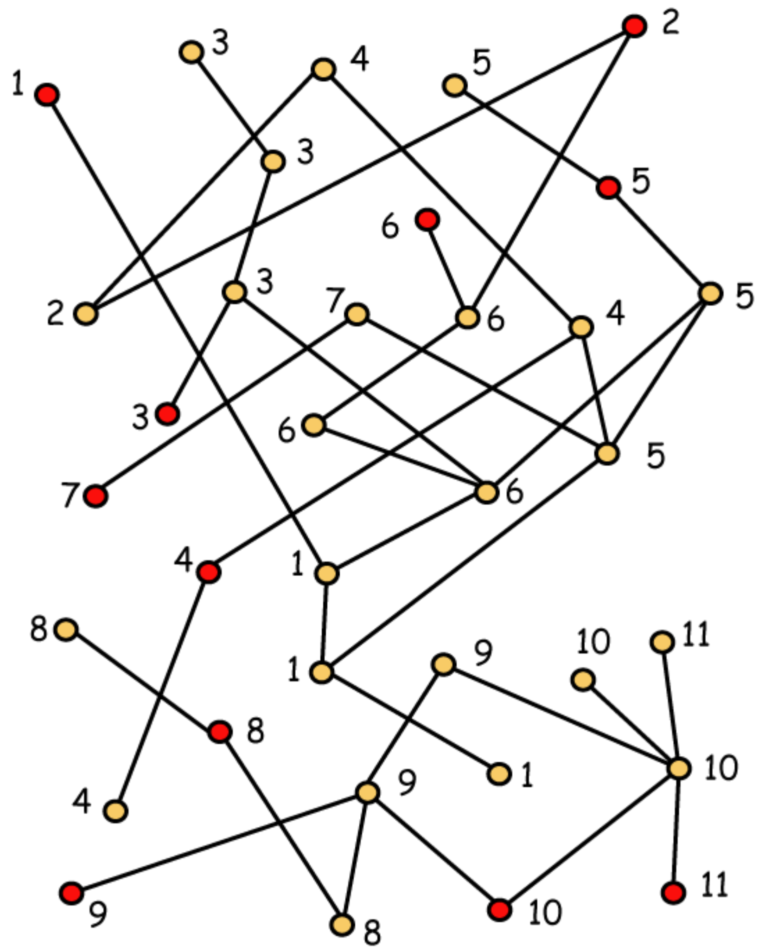
30 is a maximal element and 34 is a minimal element.

This Poset has Height 7



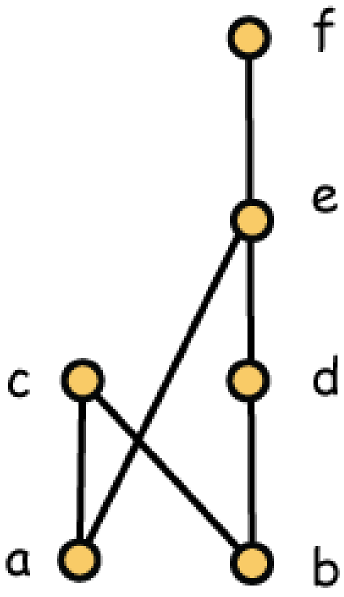
The blue points form a chain of size 7, and the coloring is a partition into 7 antichains.

And the Width is 11

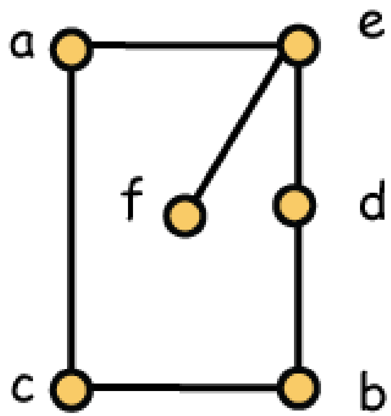


The red points form an antichain of size 11, and the coloring is a partition into 11 chains.

Diagrams and Cover Graphs

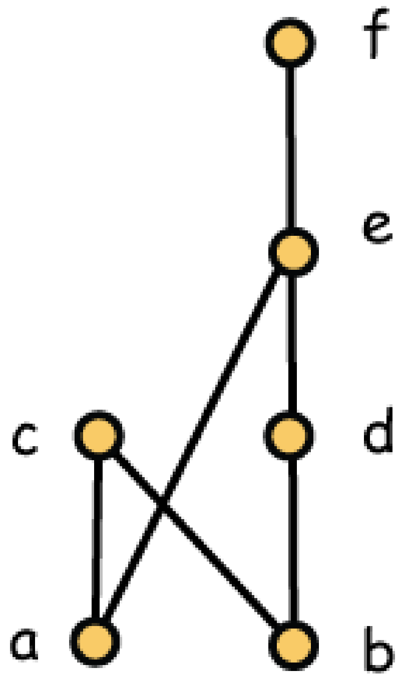


Order Diagram

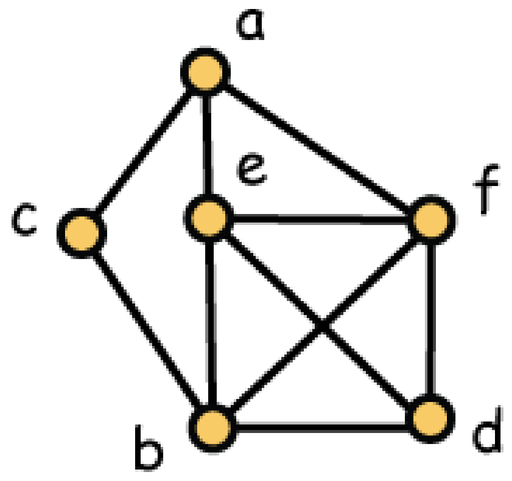


Cover Graph

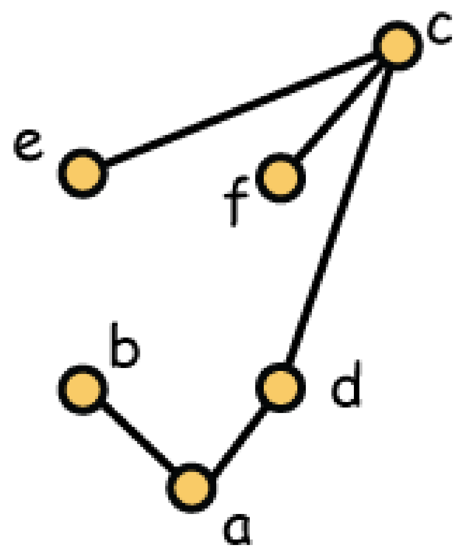
Comparability and Incomparability Graphs



Poset

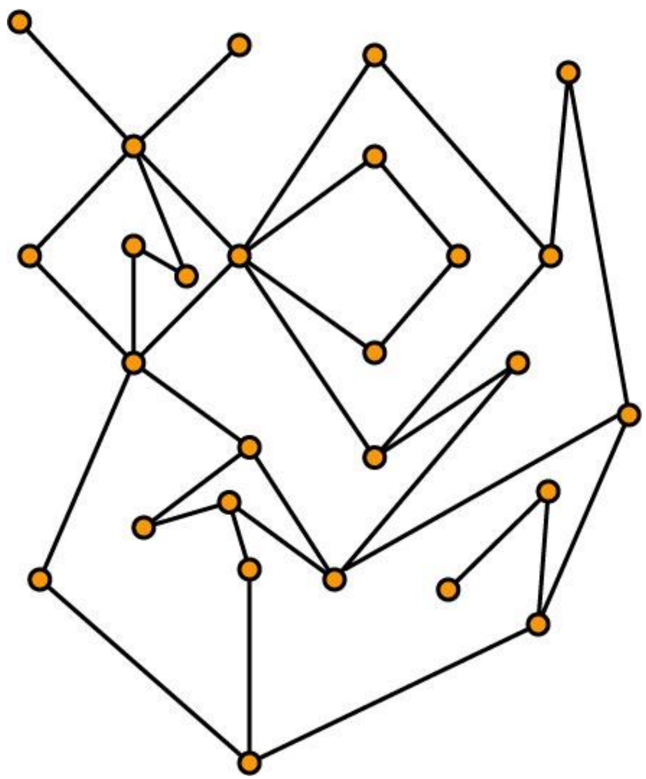


Comparability Graph



Incomparability Graph

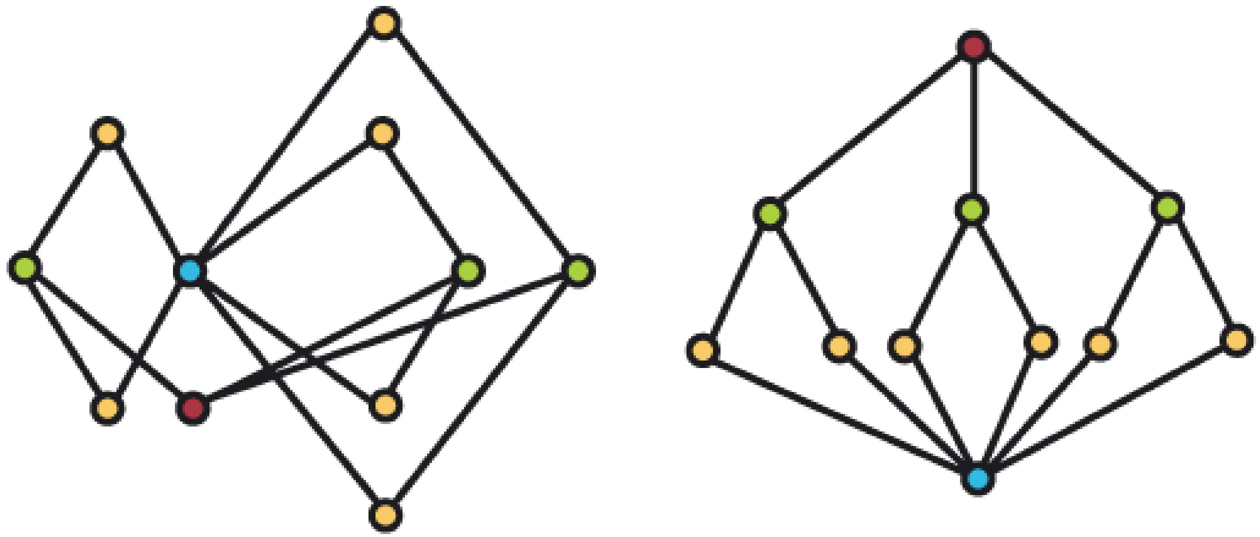
Planar Posets



Definition A poset P is planar when it has an order diagram with no edge crossings.

Fact If P is planar, then it has an order diagram with straight line edges and no crossings.

A Non-planar Poset



This height 3 non-planar poset has a planar cover graph.

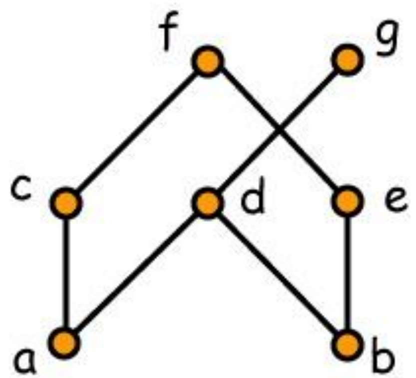
Complexity Issues

Theorem (Garg and Tamassia, '95) The question "Does P have a planar order diagram?" is NP-complete.

Theorem (Brightwell, '93) The question "Is G a cover graph?" is NP-complete.

Realizers of Posets

A family **F** = $\{L_1, L_2, \ldots, L_t\}$ of linear extensions of P is a **realizer** of P if P = $\cap$ **F**, i.e., whenever x is incomparable to y in P, there is some L_i in **F** with x > y in L_i.



$L_1 = b < e < a < d < g < c < f$

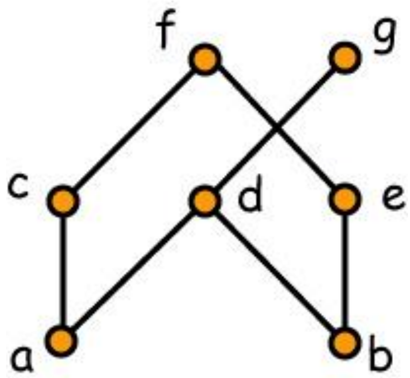
$L_2 = a < c < b < d < g < e < f$

$L_3 = a < c < b < e < f < d < g$

$L_4 = b < e < a < c < f < d < g$

$L_5 = a < b < d < g < e < c < f$

The Dimension of a Poset



$L_1 = b < e < a < d < g < c < f$

$L_2 = a < c < b < d < g < e < f$

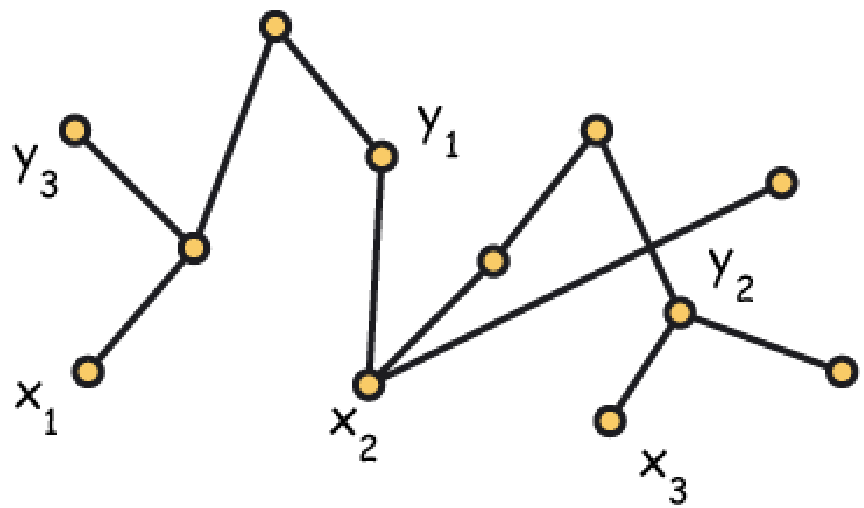
$L_3 = a < c < b < e < f < d < g$

The **dimension** of a poset is the minimum size of a realizer. This realizer shows dim(P) $\leq$ 3. In fact,

$$\dim(P) = 3$$

Dimension is Coloring for Ordered Pairs

Restatement Computing the dimension of a poset is equivalent to finding the chromatic number of a hypergraph whose vertices are the set of all ordered pairs (x, y) where x and y are incomparable in P.



Basic Properties of Dimension

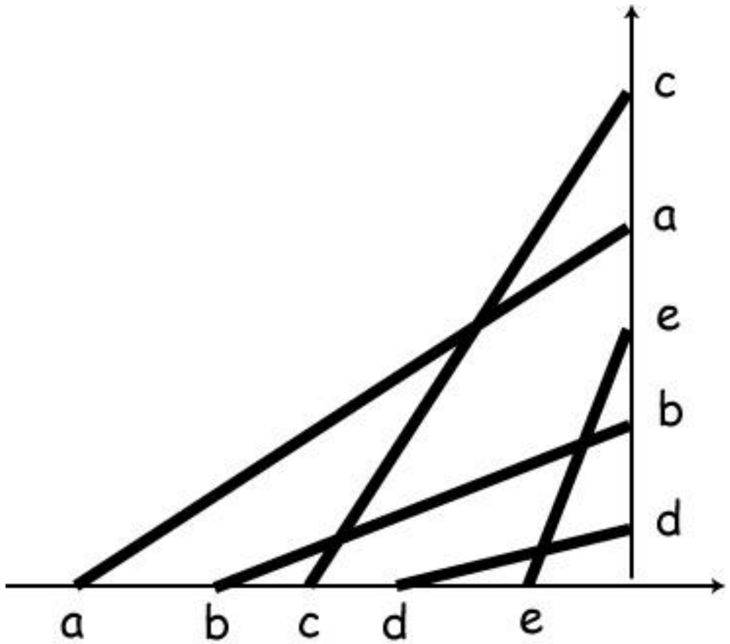
1. Dimension is monotonic, i.e., if P is contained in Q, then $\dim(P) \leq \dim(Q)$.
2. Dimension is "continuous", i.e., the removal of a point can lower the dimension by at most 1.
3. Dimension is at most the width.
4. Dimension is at most $n/2$ when P has n points and n is at least 4.

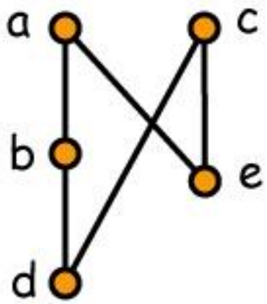
Testing $\dim(P) \leq 2$

Fact A poset P satisfies $\dim(P) \leq 2$ if and only if its incomparability graph is a comparability graph.

Fact Testing a graph on n vertices to determine whether it is a comparability graph can be done in $O(n^4)$ time.

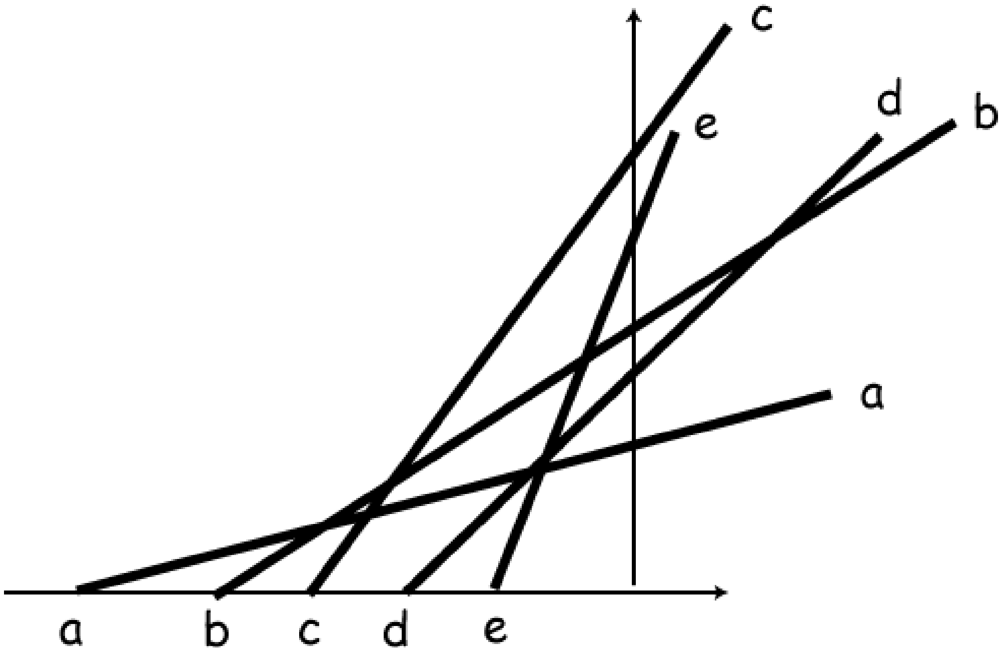
Posets of Dimension at most 2





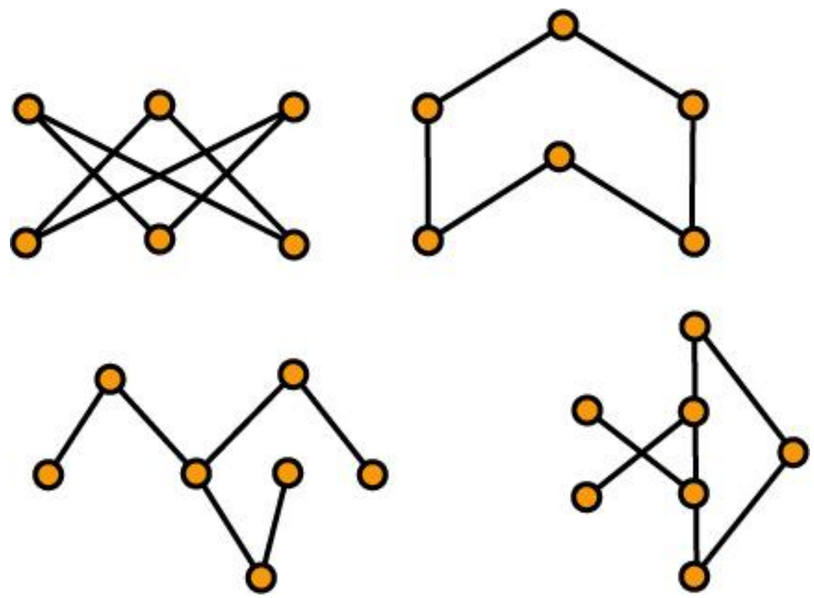
Fact A poset P has such a representation if and only if it has dimension at most 2.

A Class of Segment Orders



Talk to Csaba Biró about these fascinating objects.

3-Irreducible Posets



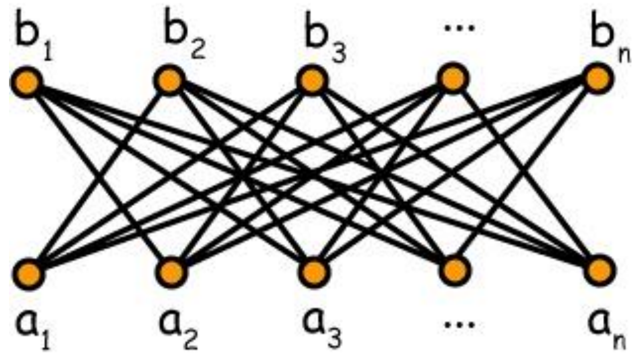
Fact These posets are irreducible and have dimension 3. The full list of all such posets is known. It consists (up to duality) of 7 infinite families and 10 other examples.

Complexity Issues for Dimension

Theorem (Yannakakis, '82) For fixed $t \geq 3$, the question $\dim(P) \leq t$? is NP-complete.

Theorem (Yannakakis, '82) For fixed $t \geq 4$, the question $\dim(P) \leq t$? is NP-complete, even when P has height 2.

Standard Examples



S_n

Fact For $n \geq 2$, the **standard example** S_n is a poset of dimension n.

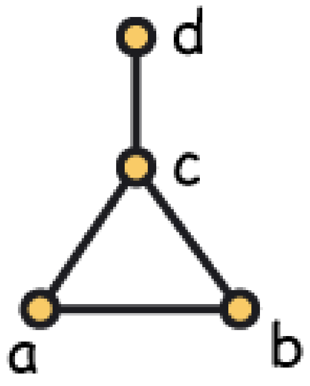
Note If L is a linear extension of S_n, there can only be one value of i for which $a_i > b_i$ in L.

Meta Question

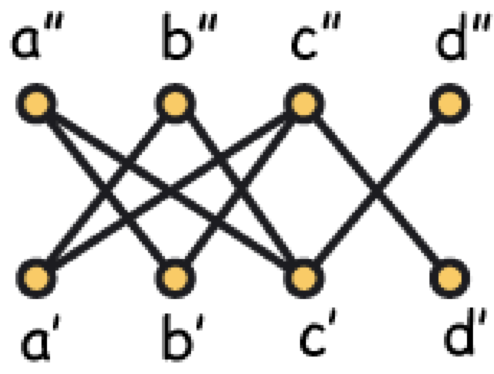
What are the combinatorial connections between graph planarity, poset planarity and parameters like height and dimension?

Adjacency Posets

The **adjacency poset** P of a graph $G = (V, E)$ is a height 2 poset with minimal elements $\{x' : x \in V\}$, maximal elements $\{x'' : x \in V\}$, and ordering: $x' < y''$ if and only if $xy \in E$.



G



P

Adjacency Posets and Dimension

Fact The standard example S_n is just the adjacency poset of the complete graph K_n.

Fact If P is the adjacency poset of a graph G, then $\dim(P) \geq \chi(G)$.

To see this, let **F** $= \{L_1, L_2, \ldots, L_t\}$ be a realizer of P. For each vertex x in P, choose an integer i with x' over x'' in L_i. This rule determines a t-coloring of G.

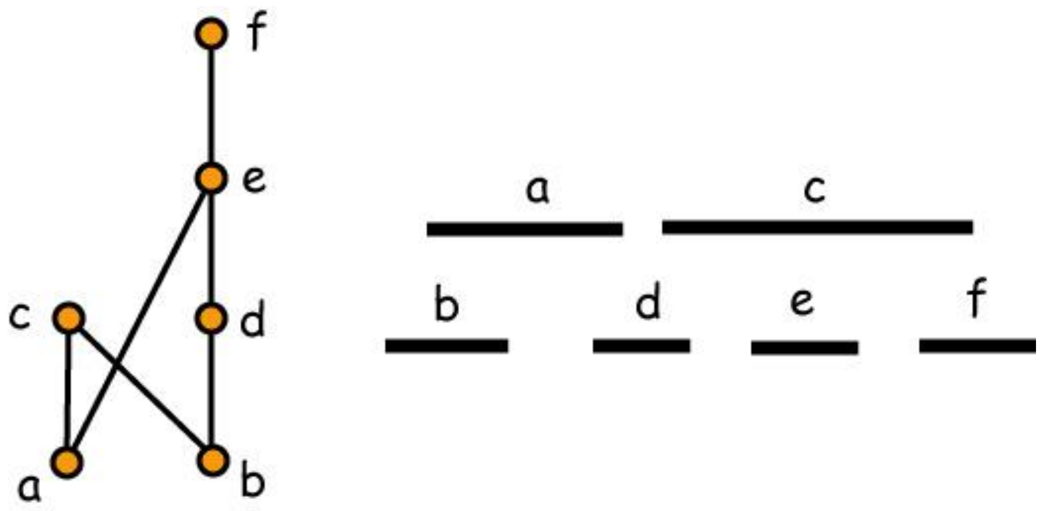
Dimension and Small Height

Theorem (Erdős, '59) For every g, t, there exists a graph G with $\chi(G) > t$ and girth of G at least g.

Observation If we take the adacency poset of such a graph, we get a poset P of height 2 for which $\dim(P) > t$ and the girth of the comparability graph of P is at least g.

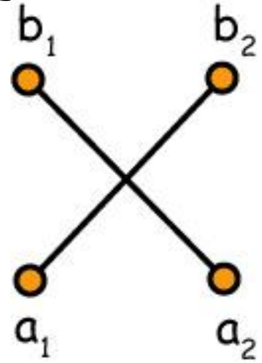
Interval Orders

A poset P is an **interval order** if there exists a function I assigning to each x in P a closed interval $I(x) = [a_x, b_x]$ of the real line $\mathbf{R}$ so that $x < y$ in P if and only if $b_x < a_y$ in $\mathbf{R}$.



Characterizing Interval Orders

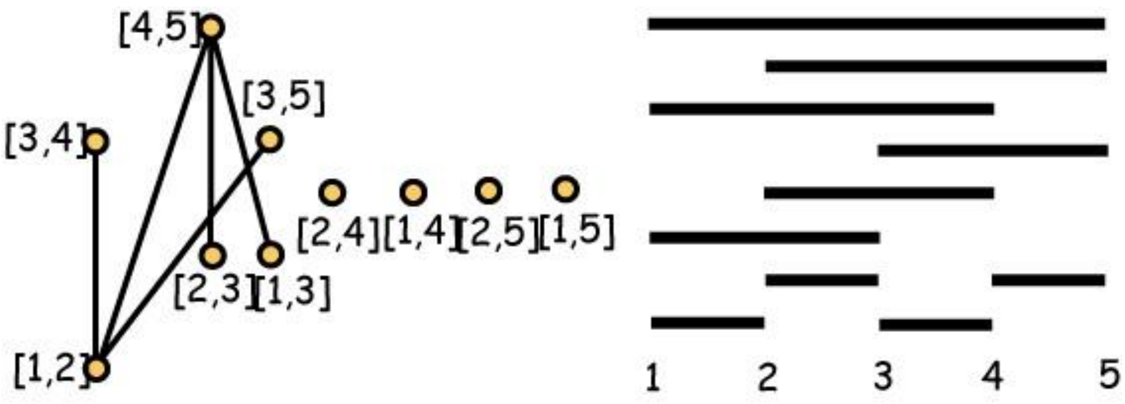
Theorem (Fishburn, '70) *A* poset is an interval order if and only if it does not contain the standard example S_2.



$$S_2 = 2 + 2$$

Canonical Interval Orders

The **canonical interval order** I_n consists of all intervals with integer end points from {1, 2, …, n}.



I_5

Dimension of Interval Orders

Theorem (Füredi, Rödl, Hajnal and WTT, '91) The dimension of the canonical interval order I_n is

$$\lg \lg n + (1/2 - o(1)) \lg \lg \lg n$$

Corollary The dimension of an interval order of height h is at most

$$\lg \lg h + (1/2 - o(1)) \lg \lg \lg h$$

Sometime Large Height is Necessary

Observation Posets of height 2 can have arbitrarily large dimension … but among the interval orders, large dimension requires large height.

The Bound is Not Tight

Fact If P is the adjacency poset of a graph G, then $\dim(P) \geq \chi(G)$.

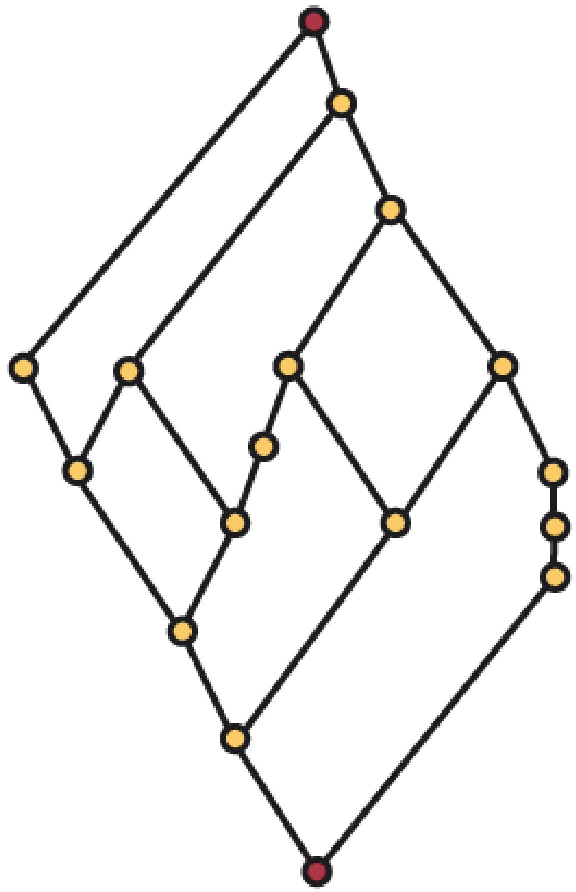
Fact If G is the subdivision of K_n, then $\chi(G) = 2$ but the dimension of the adjacency poset of G is

$$\lg \lg n + (1/2 - o(1)) \lg \lg \lg n$$

Planar Posets with Zero and One

Theorem (Baker, Fishburn and Roberts '71 + Folklore)

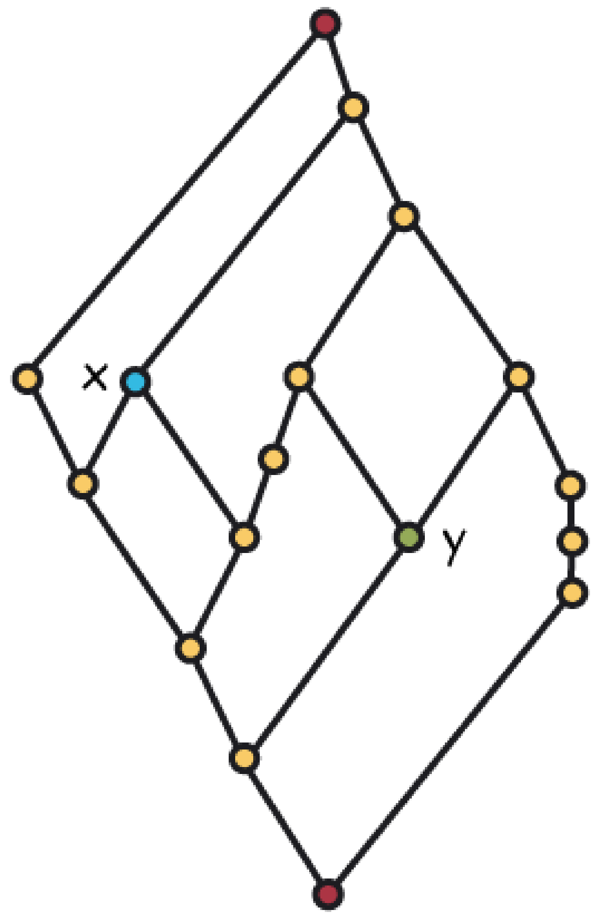
If P has both a 0 and a 1, then P is planar if and only if it is a lattice and has dimension at most 2.



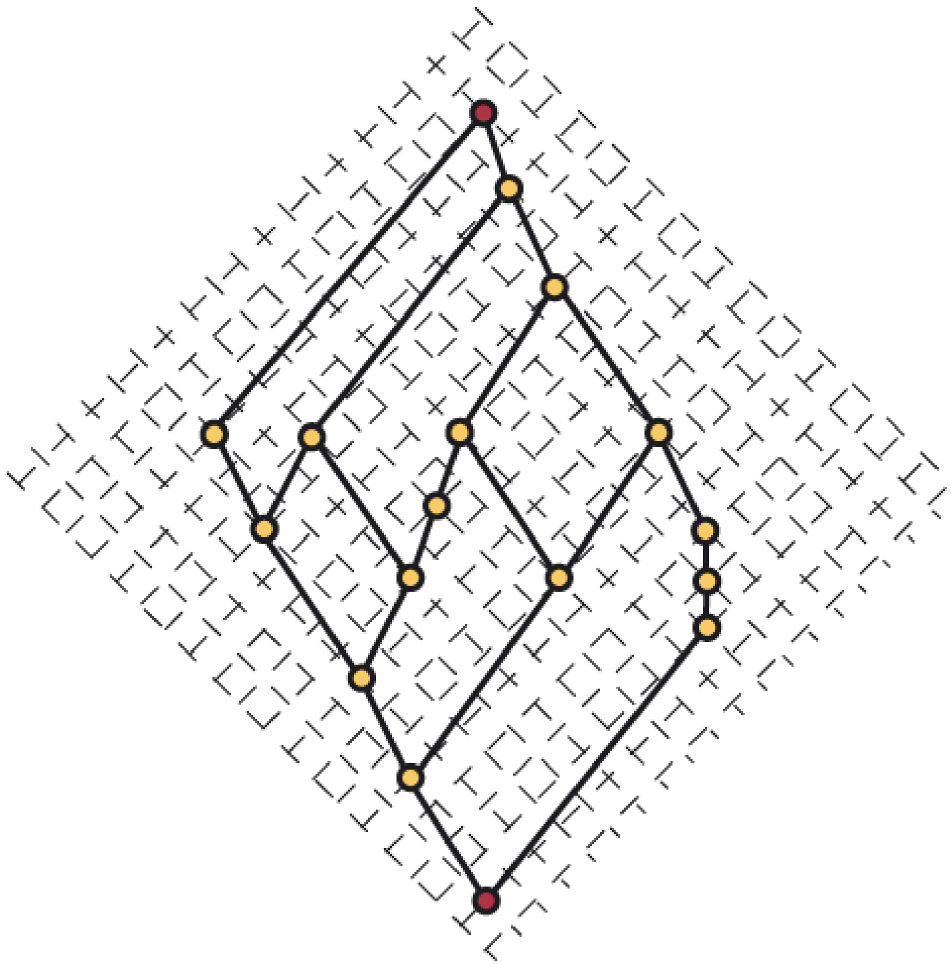
The Heart of the Proof

Observation

If x and y are incomparable, one is **left** of the other. Left is transitive.

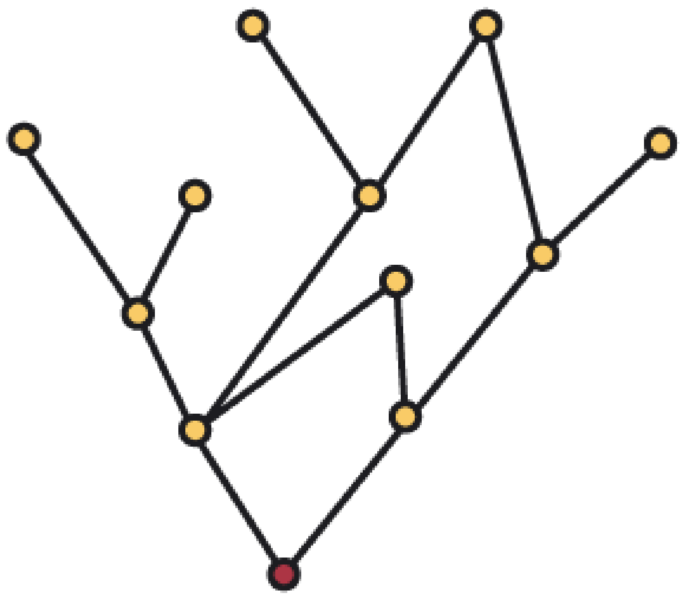


Explicit Embedding on the Integer Grid



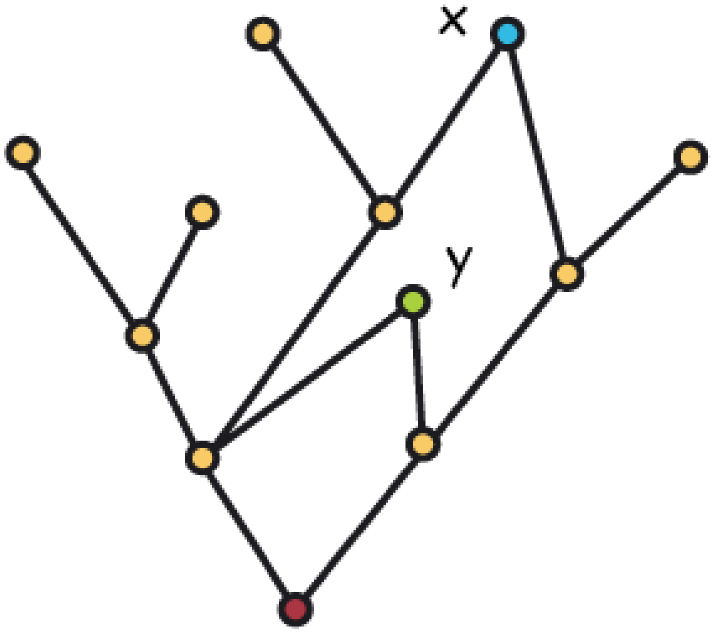
Dimension of Planar Poset with a Zero

Theorem (WTT and Moore, '77) If P has a 0 and the diagram of P is planar, then $\dim(P) \leq 3$.



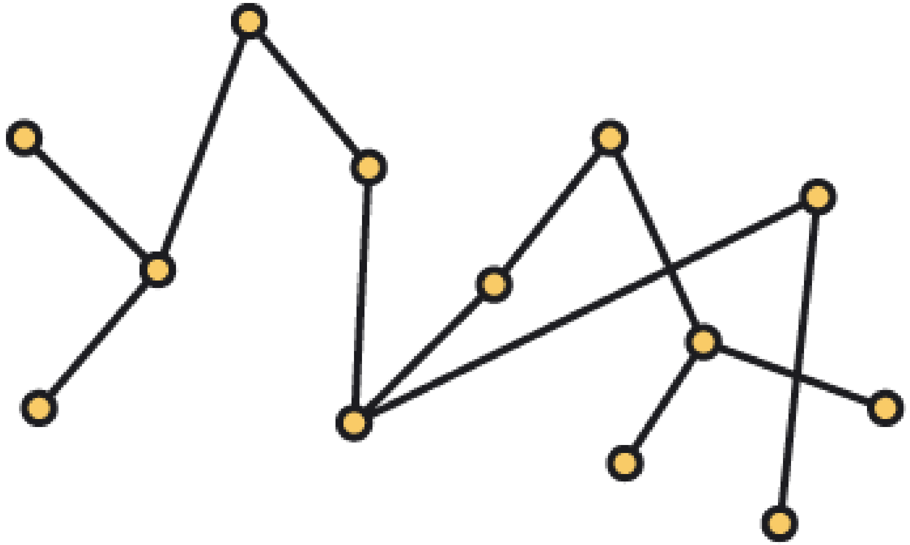
Modifying the Proof

Observation It may happen that x and y are incomparable and neither is left of the other. But in this case, one is **over** the other. Here x is over y.



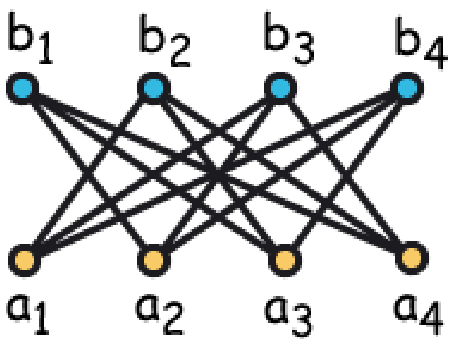
The Dimension of a Tree

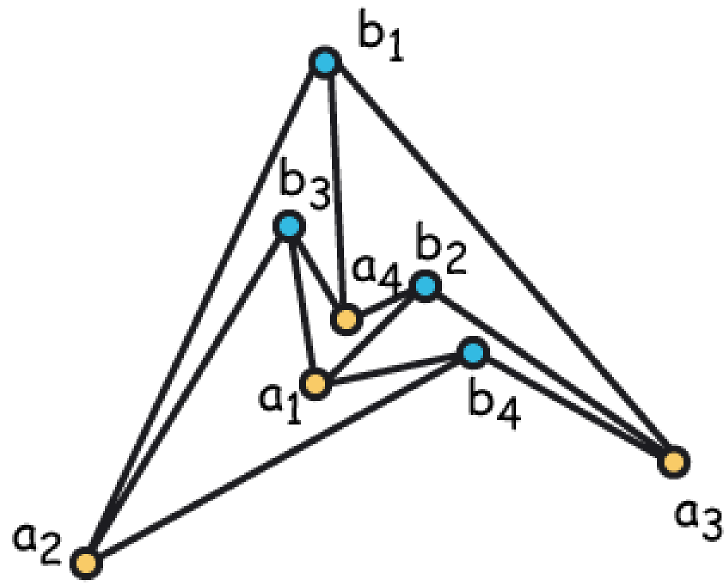
Corollary If the cover graph of P is a tree, then $\dim(P) \leq 3$.



A 4-dimensional planar poset

Fact The standard example S_4 is planar!





Wishful Thinking: If Frogs Had Wings …

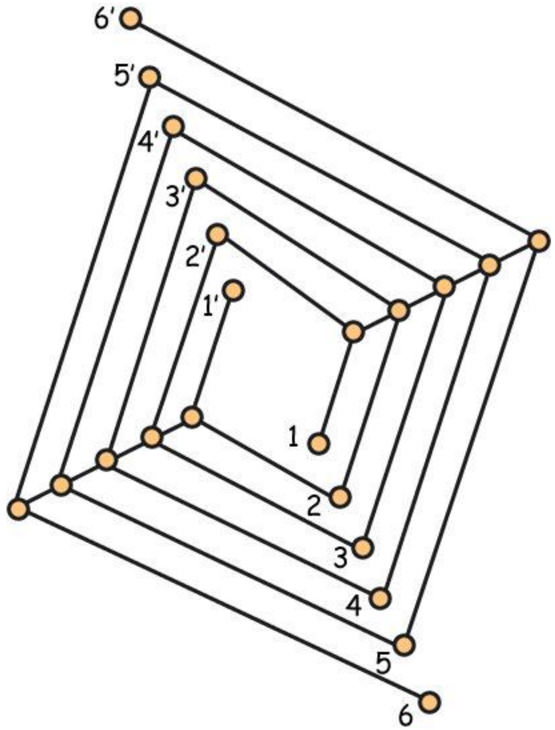
Question Could it possibly be true that dim(P) $\le$ 4 for every planar poset P?

We observe that

- dim(P) $\le$ 2 when P has a zero and a one.
- dim(P) $\le$ 3 when P has a zero or a one.
- So why not dim(P) $\le$ 4 in the general case?

No … by Kelly's Construction

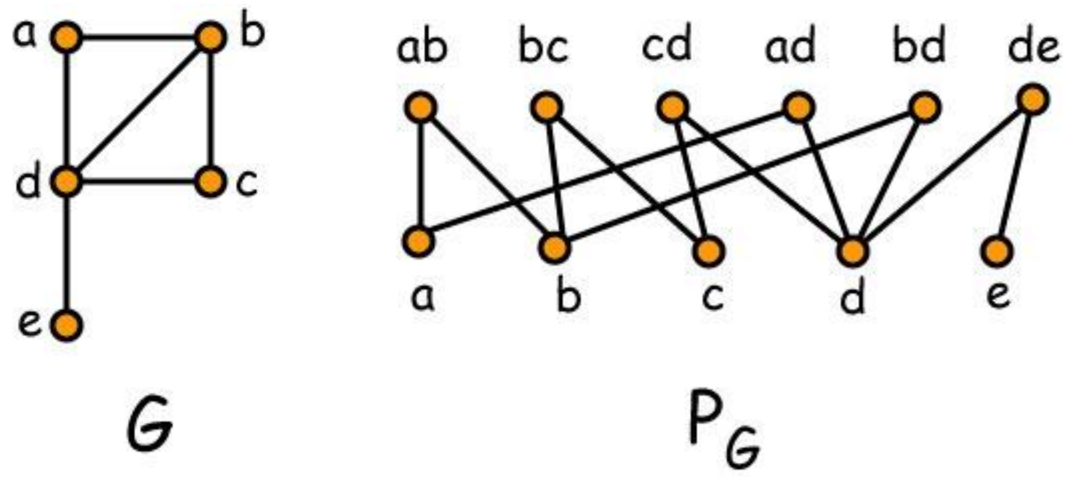
Theorem (Kelly, '81) For every $n \geq 5$, the standard example S_n is nonplanar but it is a subposet of a planar poset.



Eight Years of Silence

Kelly's construction more or less killed the subject, at least for the time being.

The Vertex-Edge Poset of a Graph



Some Elementary Observations

Fact 1 The dimension of the vertex-edge poset of K_5 is 4.

Fact 2 The dimension of the vertex-edge poset of $K_{3,3}$ is 4.

Schnyder's Theorem

Theorem (Schnyder, 89) *A graph is planar if and only if the dimension of its vertex-edge poset is at most 3.*

Note Testing graph planarity is linear in the number of edges while testing for dimension at most 3 is NP-complete!!!

The Role of Homeomorphs

Confession I didn't have the slightest idea what might be the dimension of the vertex-edge poset of a homeomorph of K_5 or $K_{3,3}$.

Timeline First contact with Schnyder was in 1986, maybe even 1985.

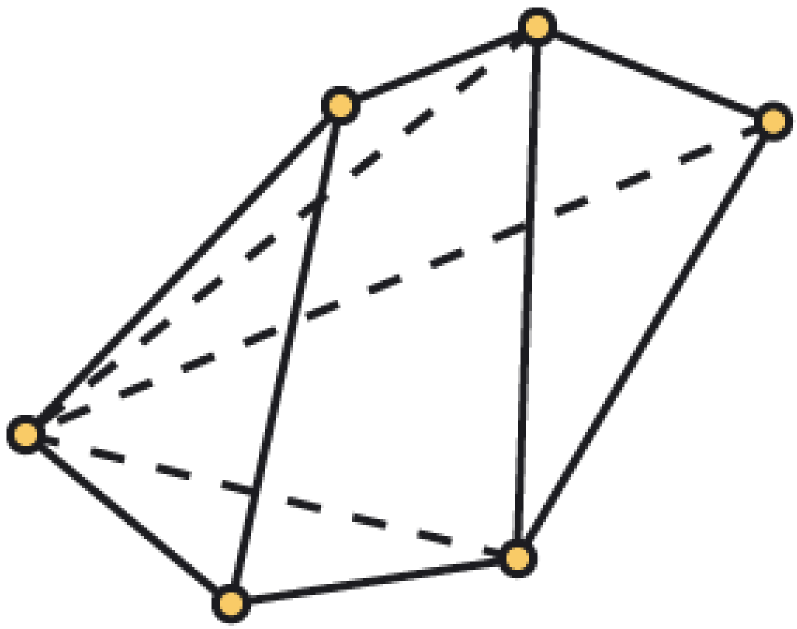
Structure and Schnyder

Schnyder's proof is a classic, elegant and rich in structure.

His motivation was to find an efficient layout of a planar graph on a small grid.

Recently, Haxell and Barrera-Cruz have found a direct proof, sans the structure, but the value of Schnyder's original approach remains intact.

Convex Polytopes and Steinitz's Theorem



3-Connected Planar Graphs

Theorem (Brightwell and WTT, '93) If G is a planar 3-connected graph and P is the vertex-edge-face poset of G, then $\dim(P) = 4$.

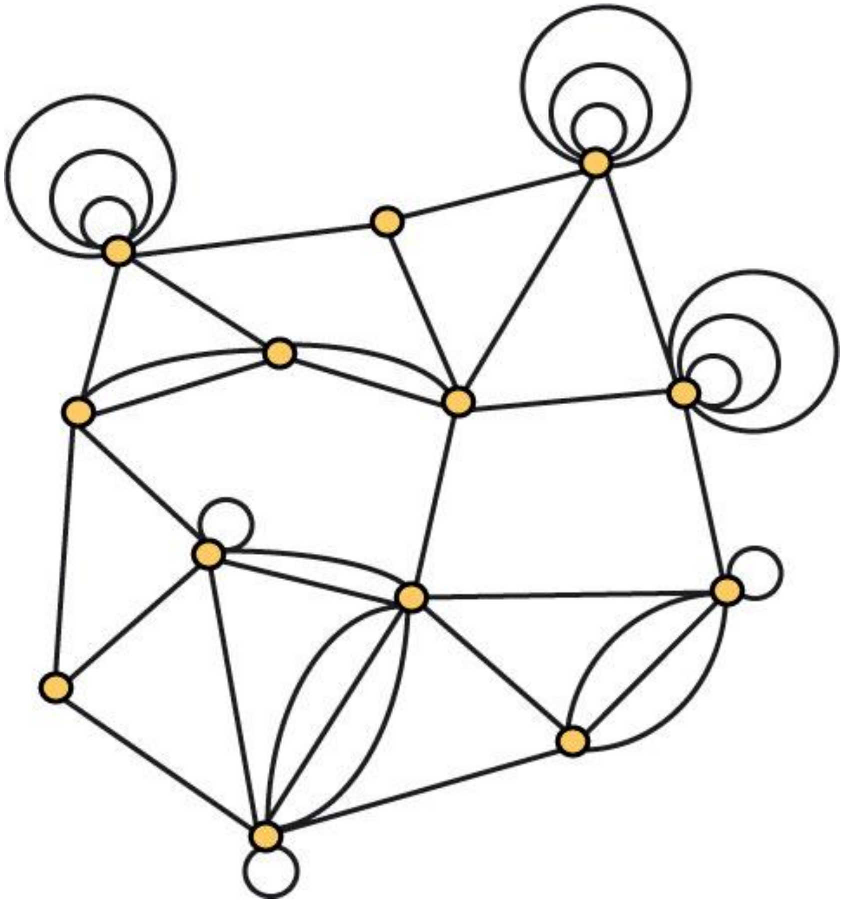
Furthermore, the removal of any vertex or any face from P reduces the dimension to 3.

Convex Polytopes

Theorem (Brightwell and WTT, '93) If M is a convex polytope in R^3, and P is the vertex-edge-face poset of M, then $\dim(P) = 4$.

Furthermore, the removal of any vertex or any face from P reduces the dimension to 3.

Planar Multigraphs



Planar Multigraphs and Dimension

Theorem (Brightwell and WTT, 97): Let D be a non-crossing drawing of a planar multigraph G, and let P be the vertex-edge-face poset determined by D. Then $\dim(P) \leq 4$.

Different drawings may determine posets with different dimensions.

Characterizing Outerplanar Graphs

Theorem (Felsner and WTT, '05) A graph is outerplanar if and only if the dimension of its vertex-edge poset is at most 5/2.

Adjacency Posets, Planarity and Genus

Fact If P is the adjacency poset of a graph G, then $\dim(P) \geq \chi(G)$ … and the inequality may be far from tight.

However, could it be true that the dimension of an adjacency poset is bounded in terms of the genus of the graph? In particular, does there exist a constant c so that $\dim(P) \leq c$ whenever P is the adjacency poset of a planar graph?

Adjacency Posets of Planar Graphs

Theorem (Felsner, Li, WTT, '10) If P is the adjacency poset of a planar graph, then dim (P) $\leq$ 8.

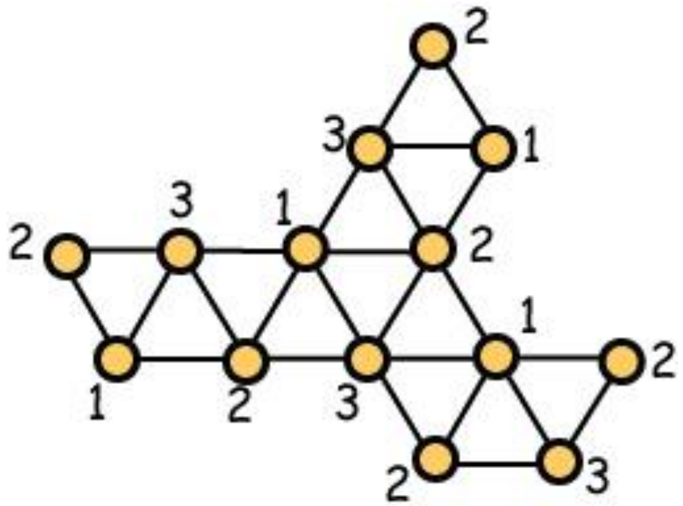
Fact There exists a planar graph whose adjacency poset has dimension 5.

Outerplanar Graphs

Theorem (Felsner, Li, WTT, '10) If P is the adjacency poset of an outerplanar graph, then $\dim(P) \leq 5$.

Fact There exists an outerplanar graph whose adjacency poset has dimension 4.

Outerplanar Graphs – Lower Bounds



Fact The dimension of the adjacency poset of this outerplanar graph is 4.

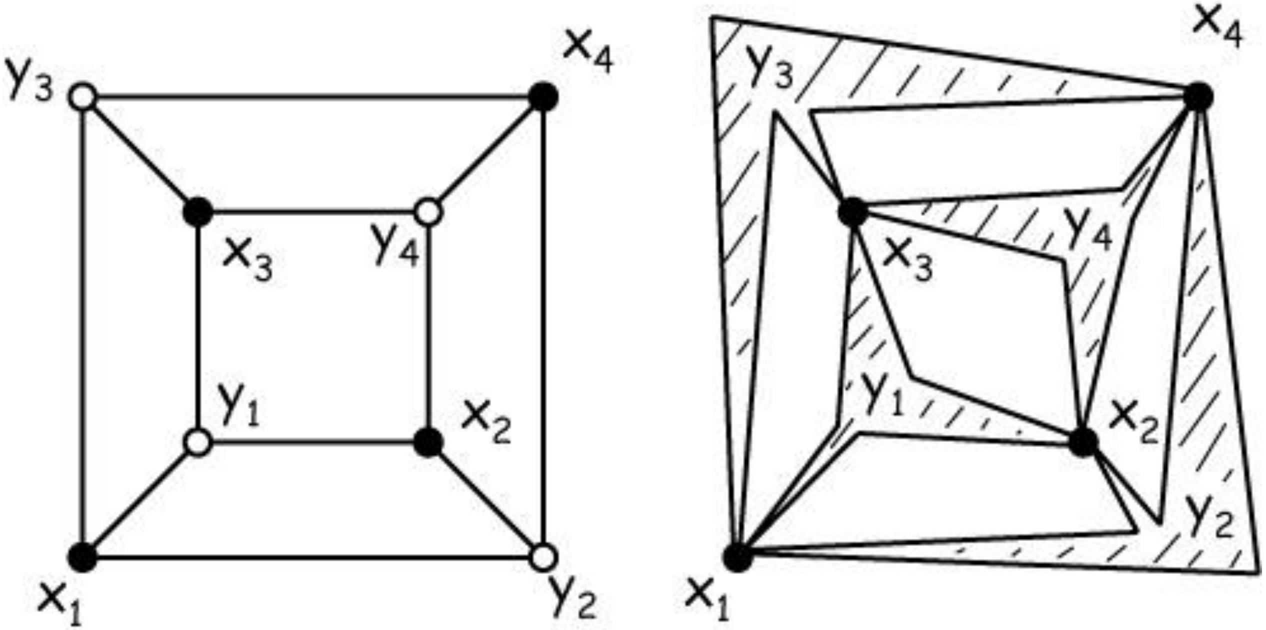
Bipartite Planar Graphs

Theorem (Felsner, Li, WTT, '10) If P is the adjacency poset of a bipartite planar graph, then $\dim(P) \leq 4$.

Corollary If P has height 2 and the cover graph of P is planar, then $\dim(P) \leq 4$.

Fact Both results are best possible.

Maximal Elements as Faces



Adjacency Posets and Genus

Theorem (Felsner, Li, WTT, '10) If the acyclic chromatic number of G is a, the dimension of the adjacency poset of G is at most $3a(a-1)/2$.

Theorem (Alon, Mohar, Sanders, '96) The acyclic chromatic number of a graph of genus g is $O(g^{4/7})$.

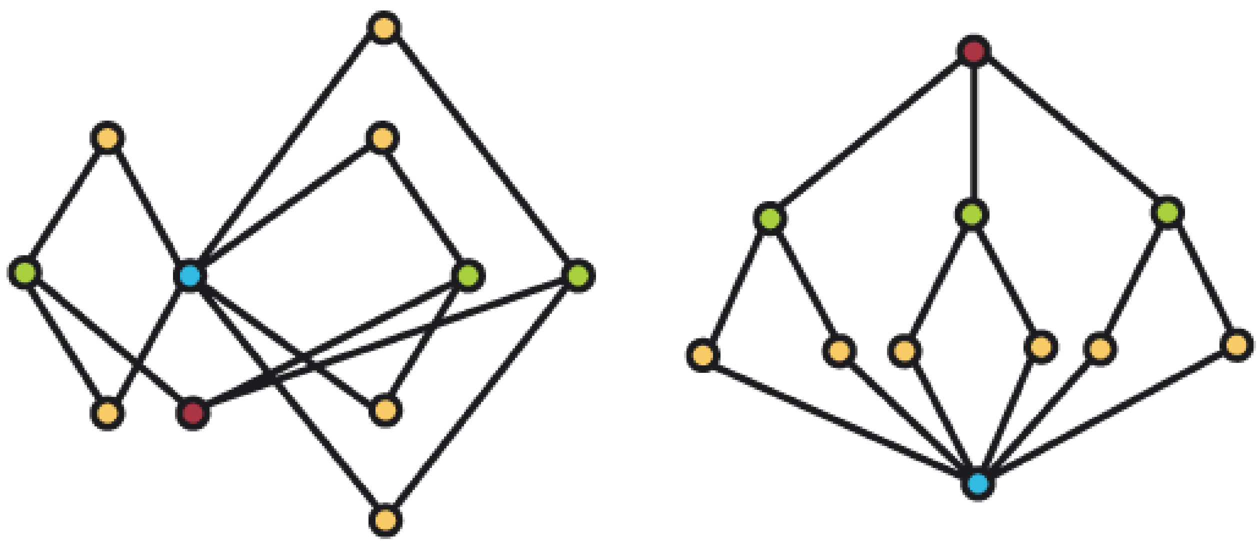
Corollary For every g, there exists a constant $c(g)$ so that if P is the adjacency poset of a graph of genus g, then $\dim(P) \leq c(g)$.

Bipartite Planar Graphs

Theorem (Moore, '72; Also Di Battista, Liu and Rival, '90) If P is a poset of height 2 and the cover graph of P is planar, then P is planar, i.e., the order diagram of P is planar.

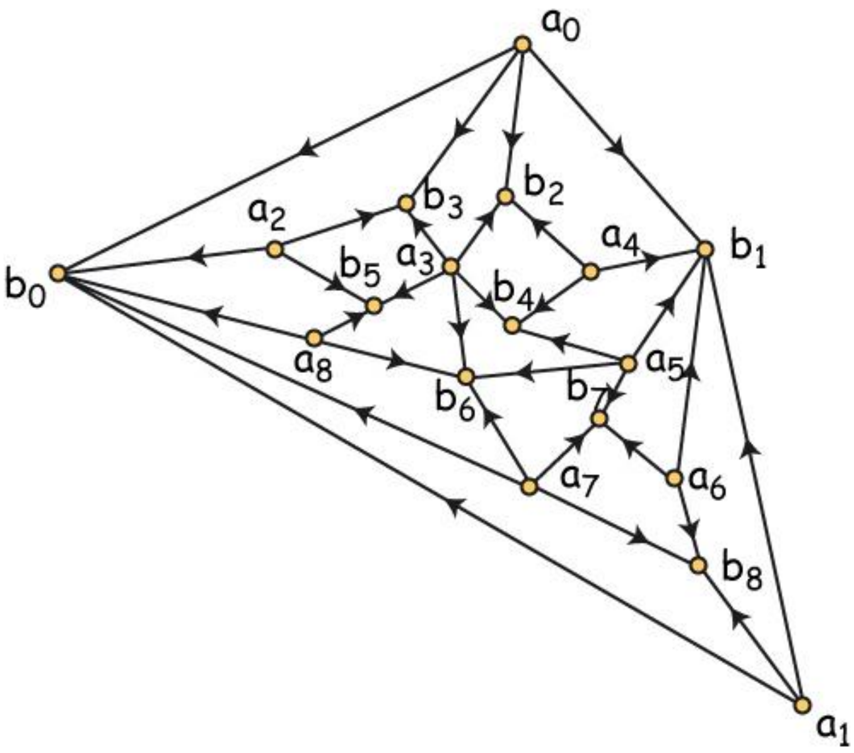
Note The result is best possible since there exist height 3 nonplanar posets that have planar cover graphs.

A Non-planar Poset



This height 3 non-planar poset has a planar cover graph.

Diagrams of Bipartite Planar Graphs



Why should it be possible to draw the order diagram of this height 2 poset without edge crossings?

Planar Cover Graphs, Dimension and Height

Conjecture (Felsner and WTT, '09) For every integer h, there exists a constant c_h so that if P is a poset of height h and the cover graph of P is planar, then $\dim(P) \leq c_h$.

Observation The conjecture holds trivially for $h = 1$ and $c_1 = 2$. Although very non-trivial, the conjecture also holds for $h = 2$, and $c_2 = 4$.

Fact Kelly's construction shows that c_h - if it exists - must be at least $h + 1$.

Conjecture Resolved

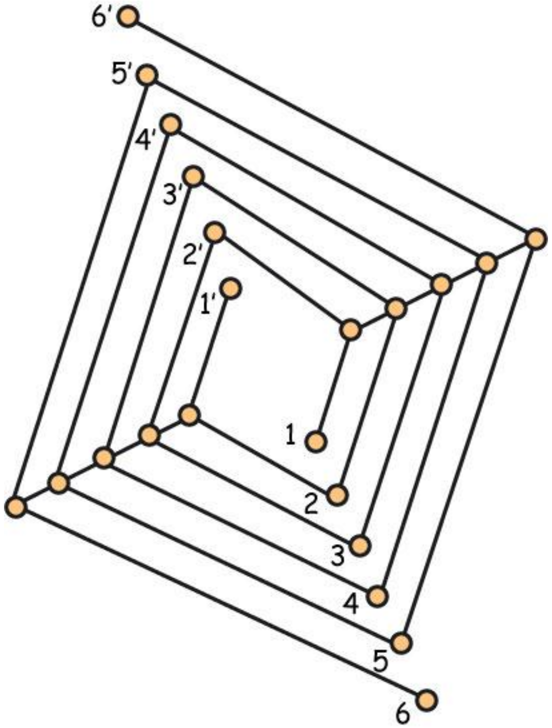
Theorem (Streib and WTT, '11) For every integer h, there exists a constant c_h so that if P is a poset of height h and the cover graph of P is planar, then $\dim(P) \leq c_h$.

Fact A straightforward modification to Kelly's construction shows that c_h must be at least $h + 2$.

However, our proof uses Ramsey theory at several key places and the bound we obtain is **very** large in terms of h.

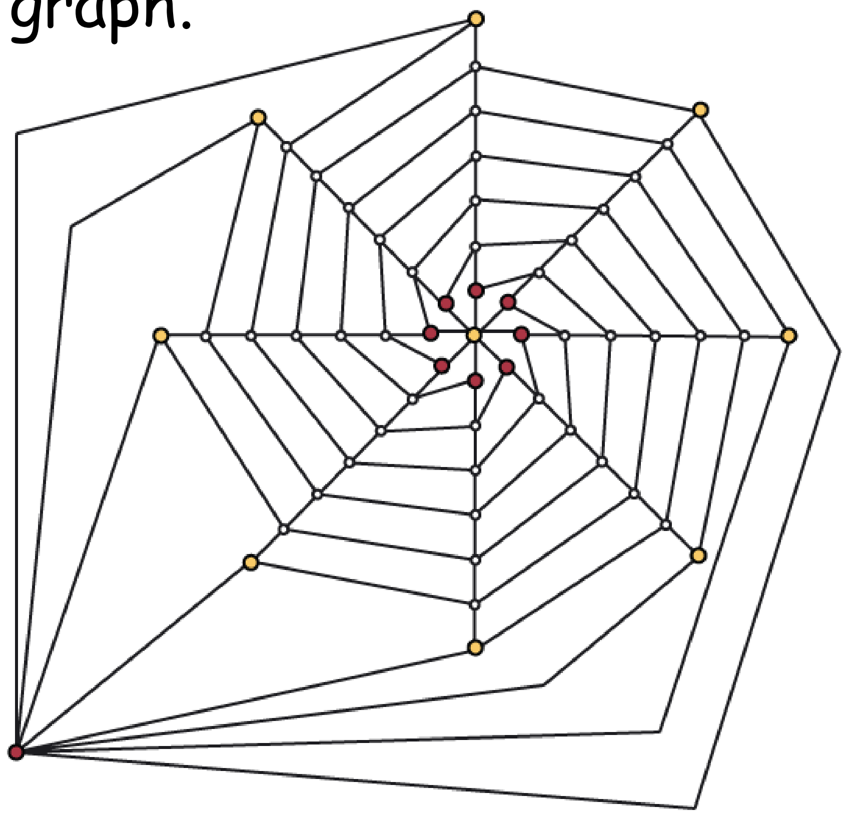
Kelly's Construction

Fact For every $h \geq 4$, the standard example S_{h-1} is contained in a planar poset of height h.



A Modest Improvement

Fact For every $h \geq 2$, the standard example S_{h+2} is contained in a poset of height h having a planar cover graph.



Some Open Questions

1. Which posets are subposets of planar posets?
2. For each $t \geq 4$, what is the smallest planar poset having dimension t?
3. Improve the bounds for the dimension of the adjacency posets of planar and outerplanar graphs.
4. Improve the bounds for the constant c_h in the Streib-WTT theorem.