Randomized Optimization Problems on Hierarchically Separated Trees

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Overview

- Some combinatorial optimization problems
- Randomized versions – history
- Hierarchically Separated Trees
- Average cost of Matching, MST, TSP
- Concentration inequalities
Goal: to minimize the total matching distance

\[ M(R, B) = \min_{\sigma} \sum d(R_i, B_{\sigma(i)}) \]
Matching on the unit square

Goal: to minimize the total matching distance

\[ M(R, B) = \min_{\sigma} \sum d(R_i, B_{\sigma(i)}) \]
Bi-chromatic Randomized Minimum Matching

\[ R, B \subset [0, 1]^d : \text{randomly, independently chosen points with } |R| = |B| = n \]

Problem: (Karp, Luby, Marchetti-Spaccamela, 1984) Find

\[ EM(R, B) = \mathbf{E} \min_\sigma \sum d(R_i, B_{\sigma(i)}) \]
The AKT Theorem

Theorem (Ajtai - Komlós - Tusnády, 1984)

Let $R, B \subset [0, 1]^2$, chosen independently, uniformly at random, such that $|R| = |B|$. Then

$$\mathbb{E}M(R, B) = \Theta(\sqrt{n \log n})$$

Other cases: (Karp, Luby, Marchetti-Spaccamela, 1984)

- $d = 1$: $\Theta(\sqrt{n})$
- $d \geq 3$: $\Theta(n^{(d-1)/d})$
More on Bi-chromatic Matchings

Shor, Leighton-Shor (1980’s):
- applications for on-line bin packing
- other models: maximal length, up-right matchings

Talagrand, Rhee-Talagrand, Talagrand-Yukich (1990’s):
- generic chaining (majorizing measures)
- applications in rectangle packing
- arbitrary norms, power weighted edges
Monochromatic Euclidean Traveling Salesman Problem

\[ X \subset [0, 1]^d, \ |X| = n, \text{ chosen independently, uniformly at random} \]

**Theorem (Beardwood, Halton, Hammersley, 1959)**

For every \( d \geq 2 \) there exists \( \alpha_d \) such that the length of the shortest tour visiting each vertex in \( X \) exactly once is

\[ \mathbb{E} TSP(X) = \alpha_d \cdot n^{(d-1)/d} \]
Monochromatic Euclidean Minimum Spanning Tree Problem

\[ X \subset [0, 1]^d, \ |X| = n, \text{ chosen independently, uniformly at random} \]

**Theorem (Steele, 1981)**

For every \( d \geq 2 \) there exists \( \beta_d \) such that the minimal total edge length of a spanning tree through \( X \) is

\[
E_{MST}(X) = \beta_d \cdot n^{(d-1)/d}
\]
Monochromatic Euclidean Minimum Matching Problem

\[ X \subset [0, 1]^d, \ |X| = n, \] chosen independently, uniformly at random

**Theorem (Avis, Davis, Steele, 1988)**

For every \( d \geq 2 \) there exists \( \gamma_d \) such that the minimal total edge length of a matching containing each vertex in \( X \) is

\[
\mathbf{E}M(X) = \gamma_d \cdot n^{(d-1)/d}
\]
Bi-chromatic Optimization Problems

Definition

\( R, B \subset [0, 1]^d, |R| = |B| = n, \) chosen independently, uniformly at random.

\( M(R, B), TSP(R, B), MST(R, B) \): Every edge in the matching, the traveling salesman tour and the spanning tree must connect two vertices with different colors.
Bi-chromatic Optimization Problems

Remark

Bi-chromatic can be much larger than monochromatic: consider matching on $[0, 1]$

*monochromatic matching*: expectation is $\approx 0.5$

*bi-chromatic matching*: expectation is $\Theta(\sqrt{n})$
Dominating Metrics

Definition
Let $\mathcal{M}_1 = (V, d_1)$, $\mathcal{M}_2 = (V, d_2)$ be finite metric spaces. $\mathcal{M}_2$ dominates $\mathcal{M}_1$ if

$$\forall x, y \in V : d_1(x, y) \leq d_2(x, y)$$

Theorem
If $\mathcal{M}_2$ dominates $\mathcal{M}_1$ then the expected total length of a functional in $\mathcal{M}_1$ is upper bounded by that of in $\mathcal{M}_2$. 
Example for domination

Subdivision of the Square

<table>
<thead>
<tr>
<th>Edge weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>0.5\sqrt{2}</td>
</tr>
<tr>
<td>0.5\sqrt{2}</td>
</tr>
<tr>
<td>0.25\sqrt{2}</td>
</tr>
</tbody>
</table>
Hierarchically Separated Trees

Yair Bartal, 1996

$\mathcal{M}(V, d)$ finite metric space
diameter $= \Delta$

leaves of the HS tree are the points of
$\mathcal{M}(V, d)$

edge weight in the $k$th level $= \Delta \cdot \lambda^k$
here $0 < \lambda < 1$

In the previous example: $\lambda = 1/2$ and $\Delta = \sqrt{2}$. 
Hierarchically Separated Trees, cont’d

**Theorem (Fakcharoenphol, Kunal, Talwar, 2003)**

Let $\mathcal{M}(V, d)$ be a finite metric space on $m$ points. Then there exists a set of dominating hierarchically separated trees, such that for any two points $x, y \in V$, if we randomly choose an HS tree from the set, the expected distance of the two points in the tree is at most $O(\log m)$ times larger:

$$E d_{HST}(x, y) = O(\log m) \cdot d(x, y)$$

**Remark:** it is necessary to use several trees - consider the case of approximating $C_m$
Hierarchically Separated Trees, cont’d

**Corollary**

Average case bounds for optimization problems on HS trees translate to good bounds for those problems in arbitrary finite metric spaces.
Monochromatic Minimum Matching

$T$ is an HST
$X$ is a randomly chosen $2n$-element sub-multiset of the leaves of $T$
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Bi-chromatic Minimum Matching

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Theorem

Let $T$ be an HST with branching factor $b$. Assume that we randomly, independently choose the $2n$-element sub-multiset $X$ of the leaves of $T$. Then

$$E M(X) = \Theta\left(\sum_{k=0}^{\delta} (b\lambda)^k\right)$$

where $\delta = \min\{\log_b n, h\}$. 
Monochromatic Traveling Salesman Problem on HST’s

**Theorem**

Let $T$ be an HST with branching factor $b$. Assume that we randomly, independently choose the $2n$-element sub-multiset $X$ of the leaves of $T$. Then

$$E_{TSP}(X) \approx 2 \cdot E_{M}(X).$$
Theorem

Let $T$ be an HST with branching factor $b$. Assume that we randomly, independently choose the $2n$-element sub-multiset $X$ of the leaves of $T$. Then

$$E_{\text{MST}}(X) \approx E_{\text{TSP}}(X)(\approx 2 \cdot E_{\text{M}}(X)).$$
Bi-chromatic Minimum Matching on HST’s

**Theorem**

Let $T$ be an HST with branching factor $b$. Assume that we randomly, independently choose two $n$-element sub-multisets $R$ and $B$ of the leaves of $T$. Then

$$EM(R, B) = \Theta(\sqrt{nb} \sum_{k=0}^{\delta} (\sqrt{b\lambda})^k),$$

where $\delta = \min\{\log_b n, h\}.$
Bi-chromatic Traveling Salesman Problem on HST’s

Theorem

Let $T$ be an HST with branching factor $b$. Assume that we randomly, independently choose two $n$-element sub-multisets $R$ and $B$ of the leaves of $T$. Then

$$\mathbb{E}_{TSP}(R, B) \approx 2 \cdot \mathbb{E}_{M}(R, B).$$
Bi-chromatic Minimum Spanning Tree on HST’s

**Theorem**

Let $T$ be an HST with branching factor $b$. Assume that we randomly, independently choose two $n$-element sub-multisets $R$ and $B$ of the leaves of $T$. Then

$$\mathbf{E}_{\text{MST}}(R, B) = \text{MST}(R \cup B) + \Theta(n \sum_{k=0}^{\delta} \lambda^{\delta-k+1} e^{-n \cdot b^{\delta-k+1}}),$$

where $\delta = \min\{\log_b n, h\}$. 
Remark

If $b \cdot \lambda > 1$ then we have a sub-gaussian behavior, that is, very tight concentration around the expected value for all monochromatic problems. These are implied by Isoperimetric Inequalities.

We don’t have this tight concentration for the bi-chromatic problems. In some cases, we managed to prove that this is not possible.
Thank you for your attention!