

# Real Analysis Qualifying Examination

Department of Mathematics

University of Louisville

January 5, 2018

This exam has two parts. Complete three problems from Part A and three problems from Part B. If you solve more than three problems in either part, clearly indicate which three you wish to have graded. Throughout the exam  $\lambda$  denotes Lebesgue measure on the line.

## Part A

1. Let  $f_n : [0, 1] \rightarrow \mathbb{R}$  be a sequence of continuously differentiable functions with the property that

- $f_k(0) = 0$  for all  $k$ , and
- $0 \leq f'_k(x) \leq f'_{k+1}(x)$  for all  $k$  and all  $x \in [0, 1]$ .

Prove that if  $\lim_{k \rightarrow \infty} f_k(1)$  exists, then  $\lim_{k \rightarrow \infty} f_k(x)$  exists for all  $x \in [0, 1]$ .

2. Prove there exist sets  $E, F \subset \mathbb{R}$  with the property that

- $\lambda(E) = 0$ ,
- $F$  is a countable union of nowhere dense sets, and
- $E \cup F = \mathbb{R}$ .

3. Let  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  be a sequence of measurable functions with the property that

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}} |f_n| d\lambda < \infty. \text{ Prove that } f_n \rightarrow 0 \text{ almost everywhere.}$$

4. Prove that if  $f : [0, 1] \rightarrow (0, \infty)$  is absolutely continuous, then  $f^2$  is absolutely continuous but  $\sqrt{f}$  may not be.

5. Prove that if  $f \in L^3(\mathbb{R})$  then  $\int_{\mathbb{R}} \frac{f(x)}{1 + |x|} dx$  exists and is finite.

## Part B

6. Let  $F(x) = \sum_{n=1}^{\infty} \frac{x^2}{x^2 + n^2}$ . Prove that  $F$  is differentiable at 0 and compute  $F'(0)$ .
7. Let  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  be a sequence of continuous functions that converges at every point  $x \in \mathbb{R}$  to a function  $f$ . Prove that  $\{x : f(x) \geq \alpha\}$  is a  $G_\delta$  set for every  $\alpha \in \mathbb{R}$ .
8. Suppose that  $f : \mathbb{R} \rightarrow (0, \infty)$  is measurable. Prove that if  $\int_A f d\lambda = 0$ , then  $\lambda(A) = 0$ .
9. Suppose that  $f : [0, 1] \rightarrow \mathbb{R}$  is absolutely continuous and that  $E \subset [0, 1]$  satisfies  $\lambda(E) = 0$ . Prove that  $\lambda(f(E)) = 0$ .
10. Decide whether or not  $L^\infty([0, 1])$  is separable and thoroughly justify your answer.

# Real Analysis Qualifier Examination

August 10, 2017

This examination has two sections. You are expected to do three problems from each section. If you submit more than three solutions from a section, indicate the three solutions you want to have graded.

In the following,  $\lambda$  is Lebesgue measure,  $\mathbb{N}$  is the natural numbers and  $\mathbb{R}$  is the real numbers. If  $f_n$  is a sequence of functions, then  $f_n \rightarrow f$  means  $f_n$  converges pointwise to  $f$  and  $f_n \rightrightarrows f$  means  $f_n$  converges uniformly to  $f$ .

## Section 1

1. If  $f \in L^p([0, \infty))$  for  $p \geq 1$ , then  $\lim_{n \rightarrow \infty} \int_0^\infty f(x)e^{-nx} dx = 0$ .
2. There is a sequence of polynomials  $p_n$  such that  $p_n(x) \rightarrow 1$  for all  $x \in [0, 1]$  and  $\int_0^1 p_n \rightarrow 2$ .
3. Suppose that  $f$  is absolutely continuous on  $[0, 1]$  and that  $f(x) \neq 0$  for all  $x \in [0, 1]$ . Prove that  $1/f$  is absolutely continuous on  $[0, 1]$ .
4. Let  $\{f_n\}$  be a sequence of Lebesgue measurable functions defined on  $[0, 1]$  with the property that  $\{f_n(x)\}$  converges almost everywhere to a function  $f$ . Prove that

$$\lim_{n \rightarrow \infty} \int_{[0,1]} \sin(f_n(x)) dx = \int_{[0,1]} \sin(f(x)) dx.$$

5. Let  $K_n$  be a sequence from  $L^1(\mathbb{R})$  satisfying
  - (a)  $\{\int_{\mathbb{R}} |K_n| : n \in \mathbb{N}\}$  is bounded,
  - (b)  $\int_{\mathbb{R}} K_n \rightarrow 1$ , and
  - (c) For all  $\delta > 0$ ,  $\int_{\{|x|>\delta\}} |K_n| \rightarrow 0$ .

If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is bounded and uniformly continuous, then  $K_n \star f \rightrightarrows f$ . (The notation  $f \star g$  is the convolution of  $f$  and  $g$ .)

## Section 2

6. Is the closed unit ball of  $L^2([0, 1])$  closed in  $L^1([0, 1])$ ? (Justify your answer.)
7. Let  $\{f_n\}$  be a sequence of real-valued Lebesgue measurable functions defined on  $\mathbb{R}$ . Define

$$E = \left\{ x \in \mathbb{R} \mid \lim_{n \rightarrow \infty} f_n(x) \text{ exists} \right\}.$$

Prove that  $E$  is Lebesgue measurable.

8. Let  $\{f_n\}$  be a sequence of continuous functions defined on  $\mathbb{R}$ . Suppose that for every  $x \in \mathbb{R}$  there exists  $n \in \mathbb{N}$  such that  $f_n(x) = 0$ . Prove that there exists an open interval  $I \in \mathbb{R}$  and an index  $n \in \mathbb{N}$  such that  $f_n(x) = 0$  for all  $x \in I$ .
9. Suppose  $K$  is a compact metric space and  $f_n : K \rightarrow \mathbb{R}$  is a sequence of continuous functions converging pointwise to a continuous function  $f$  on  $K$ . Prove that if  $\lim_{n \rightarrow \infty} f_n(x_n) = f(x)$  for every sequence  $x_n$  converging to a point  $x$ , then  $f_n \rightrightarrows f$ .
10. Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function which is Riemann integrable on  $[a, c]$  for every  $c \in (a, b)$ . Prove that  $f$  is Riemann integrable on all of  $[a, b]$ .

# Real Analysis Qualifying Exam Spring 2017

Complete **three** problems from Part A and **three** problems from Part B. Throughout the test the Lebesgue measure on  $\mathbb{R}$  is denoted by  $m$  and the Lebesgue outer measure on  $\mathbb{R}$  is denoted by  $m^*$ .

## Part A:

**Problem 1.** Prove that the series  $\sum_{k=0}^{\infty} \sin^k t$  converges uniformly for  $t \in [-\pi/4, \pi/4]$  and then evaluate the series

$$\sum_{k=0}^{\infty} \int_{-\pi/4}^{\pi/4} \sin^k t \, dt.$$

**Problem 2.** Prove that if  $A \subset \mathbb{R}$  has Lebesgue measure 0, then  $m(\{e^x : x \in A\}) = 0$ .

**Problem 3.** Let  $\{E_n\} \subset \mathcal{M}$  be a sequence of Lebesgue measurable subsets of  $[0, 1]$ . Prove:

(a) If  $\sum m(E_n) < \infty$  then  $m(\limsup E_n) = 0$

(b) If  $m(E_n) \rightarrow 0$  it may not be true that  $m(\limsup E_n) = 0$ .

**Problem 4.** Prove that if  $f : [0, 1] \rightarrow (0, \infty)$  is absolutely continuous, then so is  $1/f$ .

**Problem 5.** Prove that if  $f \in L^p([0, \infty))$ ,  $1 \leq p \leq \infty$ , then

$$\lim_{n \rightarrow \infty} \int_0^{\infty} f(x) e^{-nx} \, dx = 0.$$

## Part B:

**Problem 6.** Define the function  $f : [0, 1] \rightarrow \mathbb{R}$  by  $f(x) = 0$  if  $x$  is irrational, and by  $f(x) = \frac{1}{q}$  if  $x$  is rational and  $x = \frac{p}{q}$  when written in least terms. Decide whether or not  $f$  is Riemann integrable on  $[0, 1]$  and if so, evaluate its integral.

**Problem 7.** Let  $\{p_n\}$  be a sequence of polynomials. Suppose that for every point  $x \in [0, 1]$  there exists an index  $n$  satisfying  $p_n(x) = 0$ . Prove at least one of the polynomials is identically zero.

**Problem 8.** Let  $A \subset \mathbb{R}$ . Prove that the following are equivalent to each other:

- (a)  $A$  is not Lebesgue measurable.
- (b) There is an  $\varepsilon > 0$  such that whenever  $B$  is measurable and  $A \subset B$ , then  $m^*(B \setminus A) \geq \varepsilon$ .

**Problem 9.** Prove that if  $f$  is absolutely continuous on  $[0, 1]$  and there is a  $g \in C([0, 1])$  such that  $f' = g$  a.e., then  $f$  is differentiable on  $[0, 1]$  and  $f' = g$ .

**Problem 10.** Let  $h \in L^\infty(\mathbb{R})$ . Define a functional  $T : L^1(\mathbb{R}) \rightarrow \mathbb{R}$  by

$$Tf = \int_{\mathbb{R}} fh \, dm$$

Prove  $\sup_{\|f\|_1 \leq 1} Tf = \|h\|_\infty$ .

# Analysis Qualifying Examination

Department of Mathematics  
University of Louisville  
August 11, 2016

This test has two sections. Do four problems from each section. If you do more than four problems, indicate the solutions to be graded.

## Section A

1. Let  $C \subset [0, 1]$  be a closed set. Prove that  $\chi_C$  is Riemann integrable if and only if  $\partial C$  has Lebesgue measure zero.

2. Let  $S \subset \mathbb{R}$ . Prove the following statements are equivalent to each other:

(a)  $S$  is Lebesgue measurable.

(b) There is a  $G_\delta$  set  $G$  and a set  $N$  of measure zero such that  $S = G \setminus N$ .

3. If  $f$  is nonnegative and integrable on  $[0, 1]$ , then

$$\lim_{n \rightarrow \infty} \int_0^1 \sqrt[n]{f} = \lambda\{x : f(x) > 0\}.$$

4. Let  $f \in L^1(\mathbb{R})$ . If  $\int_a^b f = 0$  for all rational numbers  $a$  and  $b$  with  $a < b$ , then  $f = 0$  a.e..

5. Suppose that  $f \in L^2([0, 1])$  and  $\|f\|_2 = 1$ . Define  $g(x) = xf(x)$ . Prove that  $g \in L^1([0, 1])$  and that  $\|g\|_1 \leq \frac{1}{\sqrt{3}}$ .

## Section B

6. Let  $\{a_k\}$  be a sequence of real numbers with the property that  $|a_k| \leq 1$  for all  $k$ . Prove that both series

$$f(x) = \sum_{k=1}^{\infty} a_k x^k, \quad g(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

converge uniformly on every compact subinterval of  $(-1, 1)$  and that  $f'(x) = g(x)$  for all  $x \in (-1, 1)$ .

7. Give an example of a continuous function  $f : [0, 1] \rightarrow \mathbb{R}$  with the property that  $f(0) = 0$ ,  $f(1) = 1$ , yet  $f'(x) \leq -1$  for almost every  $x \in [0, 1]$ .

8. Let  $E_1, E_2, E_3, \dots$  be a sequence of measurable subsets of  $\mathbb{R}$  with the property that  $\sum_{n=1}^{\infty} \lambda(E_n) < \infty$ . Show almost every  $x \in \mathbb{R}$  is contained in only finitely many of the  $E_n$ .

9. Let  $f : [0, 1] \rightarrow \mathbb{R}$  be Lebesgue measurable. Prove that if

$$p \leq f(x) \leq q$$

for all  $x \in [0, 1]$ , then  $\int_{[0,1]} f$  exists and

$$p \leq \int_{[0,1]} f \leq q.$$

10. Define a sequence of functions  $f_n \in L^1[0, 1]$  by

$$f_n(x) = \begin{cases} n, & x \leq 1/n \\ 0, & x > 1/n \end{cases}.$$

Does  $f_n$  converge in  $L^1[0, 1]$ ? If so, to what function?

11. Let  $H$  be a real Hilbert space. Prove that  $\langle x, y \rangle = 0$  if and only if  $\|x\| \leq \|x + \lambda y\|$  for every  $\lambda \in \mathbb{R}$ .



# Analysis Qualifying Examination

Department of Mathematics  
University of Louisville  
January 2016, 9:00am–12:30pm

This test has two sections. Do three problems from each section. If you do more than three problems, indicate the solutions to be graded.

## Section A

1. Let  $S$  be dense in  $\mathbb{R}$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Prove or give a counterexample:  $f$  is measurable if and only if  $\{x : f(x) \geq s\}$  is measurable for all  $x \in S$ .
2. Suppose  $\lambda(S)$  denotes the Lebesgue measure of the set  $S \subset \mathbb{R}$ . Let  $g : [0, 1] \rightarrow \mathbb{R}$  be absolutely continuous and  $E \subset [0, 1]$  be such that  $\lambda(E) = 0$ . Prove that  $\lambda(g(E)) = 0$ .
3. Let  $f_n(x) = x^n$  for each  $n \geq 1$ . Prove that the sequence  $\{f_n\}$  converges uniformly on  $[-\delta, \delta]$  for each  $0 < \delta < 1$ , and converges non-uniformly on  $(-1, 1)$ .
4. Let  $\lambda(G)$  denote the Lebesgue measure of the set  $G$ . Find an open set  $G$  which is dense in  $[0, 1]$  such that  $\lambda(G) < 1$  and  $\lambda(G \cap I) > 0$  for any interval  $I \subset [0, 1]$ .

## Section B

5. Is  $L^p([a, b])$  separable, where  $1 < p < \infty$ ?

6. Suppose that  $1 < p, q < \infty$  and that  $\frac{1}{p} + \frac{1}{q} = 1$ . Prove that if  $f_n \rightarrow f$  in  $L^p(\mathbb{R})$  and  $g_n \rightarrow g$  in  $L^q(\mathbb{R})$ , then  $f_n g_n \rightarrow fg$  in  $L^1(\mathbb{R})$ .

7. Let  $f \in L^1([0, 1])$ . Prove that

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) \cos nx \, dx = 0.$$

8. Assume that  $f \in L^\infty([0, 1])$ . Prove that  $f \in L^p([0, 1])$  for each  $1 \leq p < \infty$  and that  $\|f\|_\infty = \lim_{p \rightarrow \infty} \|f\|_p$ .

# Analysis Qualifying Examination

Department of Mathematics

University of Louisville

August 13, 2015, 9:00am–12:30pm

This test has two sections. Do three problems from each section. If you do more than three problems, indicate the solutions to be graded.

## Section A

1. Let  $C \subset \mathbb{R}$  denote the Cantor set. Let  $\chi_C(x) = 1$  if  $x \in C$  and 0 otherwise. Explain why  $\chi_C$  is Riemann integrable and compute  $\int_0^1 \chi_C(x) dx$ .

2. Let  $E \subset \mathbb{R}$  be a Lebesgue measurable set with  $m(E) = 1$ . Prove there exists a Lebesgue measurable set  $F \subset E$  with  $m(F) = \frac{1}{2}$ .

3. Let  $(X, \mathcal{A}, \mu)$  be a measure space. If  $f_n : X \rightarrow \mathbb{R}$  is a sequence of functions such that  $\sum_{n=1}^{\infty} \int_X |f_n| d\mu$  converges, then prove that  $f_n \rightarrow 0$  almost everywhere.

4. Prove that

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \cos\left(\frac{1}{x^2}\right) & \text{if } x \neq 0 \end{cases}$$

is continuous but not absolutely continuous on  $[-1, 1]$ .

5. Let  $(X, \mathcal{A}, \mu)$  be a finite measure space. If  $f$  is  $\mu$ -measurable and

$$p \leq f(x) \leq q$$

for all  $x \in X$ , then prove that  $\int_X f d\mu$  exists and

$$p\mu(X) \leq \int_X f d\mu \leq q\mu(X).$$

## Section B

6. Suppose that  $1 < p, q < \infty$  and that  $\frac{1}{p} + \frac{1}{q} = 1$ . Prove that if  $f_n \rightarrow f$  in  $L^p(\mathbb{R})$  and  $g_n \rightarrow g$  in  $L^q(\mathbb{R})$ , then  $f_n g_n \rightarrow fg$  in  $L^1(\mathbb{R})$ .

7. Evaluate  $\frac{d}{dt} \int_0^1 \frac{\sin(xt)}{x} dx$ . Justify your computations.

8. Let  $H$  be a Hilbert space. Prove that if  $\{x_\alpha\}_{\alpha \in A}$  is an orthonormal set in  $H$  then

$$\sum_{\alpha \in A} |\langle x, x_\alpha \rangle|^2 \leq \|x\|^2$$

for all  $x \in H$ .

9. If  $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$  and  $p \geq 1$ , then prove that  $f \in L^p(\mathbb{R})$ .

10. If  $f : \mathbb{R} \rightarrow [0, \infty)$  is measurable, then  $\lim_{n \rightarrow \infty} \int_{-n}^n f = \int_{\mathbb{R}} f$ .

# ANALYSIS QUALIFIER EXAM

JANUARY 2013

The Lebesgue measurable subsets of  $\mathbb{R}$  are denoted by  $\mathcal{L}$  and Lebesgue measure is denoted by  $\lambda$ .

This test has two sections and two pages with five problems in each section. You are expected to do three problems from each section. Solutions to at most three problems from each section will be graded. If you do more than three problems, indicate the solutions to be graded.

## SECTION A

**Problem 1.** Show that every dense subset of  $L^\infty([0, 1])$  is uncountable.

**Problem 2.** Let  $f$  be a Lebesgue measurable function on  $\mathbb{R}$  with the property that

$$\sup_{\{g \in L^2(\mathbb{R}) : \|g\|_2 \leq 1\}} \int_{\mathbb{R}} |fg| d\lambda \leq 1.$$

Prove that  $f \in L^2(\mathbb{R})$  and  $\|f\|_2 \leq 1$ .

**Problem 3.** Let  $f \geq 0$  and  $f \in L^p[0, 1]$  for all  $p \in [1, \infty)$ . If  $\|f\|_p^p = \|f\|_1$  for all  $p \in [1, \infty)$ , then there is a set  $S$  such that  $f = \chi_S$  a.e.

**Problem 4.** If  $E$  is a measurable subset of  $\mathbb{R}$ , then there is an interval  $I$  such that  $\lambda(E \cap I) > \frac{9}{10}\lambda(I)$  or  $\lambda(E^c \cap I) > \frac{9}{10}\lambda(I)$ .

**Problem 5.** A measure space  $(X, \mu)$  is  $\sigma$ -finite iff there is an  $f : X \rightarrow (0, \infty)$  such that  $f \in L^1(X, \mu)$ .

## SECTION B

**Problem 6.** (a) Find a sequence  $f_n : [0, 1] \rightarrow \mathbb{R}$  such that  $\int_0^1 |f_n(x)| = 2$  for all  $n \in \mathbb{N}$  and

$$\lim_{n \rightarrow \infty} f_n(x) = 1, \quad \forall x \in [0, 1].$$

(b) If the  $f_n$  are as in part (a), then prove

$$\lim_{n \rightarrow \infty} \int_0^1 |f_n(x) - 1| dx = 1.$$

**Problem 7.** Show that  $\mathcal{G} = \{f \in C[0, 1] : \int_0^1 f^2 > 1\}$  is open in  $C[0, 1]$ . (Assume  $C[0, 1]$  has the uniform metric.)

**Problem 8.** Let  $(X, \rho)$  be a metric space and suppose  $K$  and  $F$  are nonempty disjoint subsets of  $X$  with  $K$  compact and  $F$  closed.

- (a) Prove there is a  $\delta > 0$  such that  $\rho(x, y) \geq \delta$  for all  $x \in K$  and  $y \in F$ .  
 (b) Show that part (a) may fail if  $K$  is closed, but not compact.

**Problem 9.** The limit superior of a sequence of sets  $\{E_k\}$  is defined as

$$\limsup E_k = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_k.$$

Let  $\{E_k : k \in \mathbb{N}\}$  be a sequence of sets in  $\mathcal{L}$ .

- (a) Prove that if  $\sum_{k \in \mathbb{N}} \lambda(E_k) < \infty$ , then  $\lambda(\limsup E_k) = 0$ .  
 (b) Is it true in general that  $\lambda(\limsup E_k) = \limsup \lambda(E_k)$ ?

**Problem 10.** Show that

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is in  $BV[-1, 1]$ , but

$$g(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is not.

# Analysis Qualifying Exam August 9, 2010

This exam consists of three parts. For each part work the number of problems indicated. If you work more problems, clearly indicate which should be graded. Please write only on one side of each sheet of paper and put your name in the upper right hand corner. Lebesgue measure is denoted by  $\lambda$ .

## PART I (do any 3 problems)

**Problem 1:** Suppose that  $\{f_n\}$  is a sequence of nonnegative Lebesgue integrable functions on  $\mathbb{R}$  such that  $f_n \rightarrow f$  almost everywhere, with  $f$  Lebesgue integrable, and that  $\int_{\mathbb{R}} f_n d\lambda \rightarrow \int_{\mathbb{R}} f d\lambda$ . Prove that  $\int_{\mathbb{R}} |f_n - f| d\lambda \rightarrow 0$ .

**Problem 2:** Suppose  $f$  is absolutely continuous on  $\mathbb{R}$ ,  $f \in L^1(\mathbb{R})$  and in addition that

$$\lim_{t \rightarrow 0^+} \int_{\mathbb{R}} \left| \frac{f(x+t) - f(x)}{t} \right| dx = 0$$

show that  $f = 0$ .

**Problem 3:** Let  $f \in L^1([a, b])$  and  $\{g_n\}$  be a sequence of bounded Lebesgue measurable functions defined on  $[a, b]$  which converges uniformly to  $g$ . Show that

$$\lim_{n \rightarrow \infty} \int_{[a, b]} f g_n d\lambda = \int_{[a, b]} f g d\lambda.$$

**Problem 4:** a) Give an example of a sequence of Lebesgue integrable functions  $\{f_n\}$  such that  $f_n(x) \rightarrow f(x)$  for all  $x \in \mathbb{R}$ , where  $f \in L^1$ , but  $\int_{\mathbb{R}} f_n d\lambda \not\rightarrow \int_{\mathbb{R}} f d\lambda$ .  
b) Give an example of a sequence  $\{f_n\}$  of nonnegative and uniformly bounded functions on  $[0, 1]$  such that  $\lim_{n \rightarrow \infty} \int_{[0, 1]} f_n d\lambda$  exists, but  $f_n(x)$  diverges for all  $x \in [0, 1]$ .

## PART II (do any 2 problems)

**Problem 5:** Show that  $A \subseteq \mathbb{R}$  has Lebesgue measure 0 if and only if there is a countable collection of open intervals  $J_n$  such that

a)  $\sum_{n=1}^{\infty} \lambda(J_n) < \infty$

b) for every  $x \in A$  the set  $\{n \in \mathbb{N} | x \in J_n\}$  is infinite, i.e. every point of  $A$  is in infinitely many of the sets  $J_n$ .

**Problem 6:** Let  $\{A_n\}$  be a sequence of measurable subsets of  $\mathbb{R}$ . Show that if there is a  $K \in \mathbb{N}$  such that  $\lambda(\bigcup_{n \geq K} A_n) < \infty$  then

$$\lambda\left(\limsup_{n \rightarrow \infty} A_n\right) \geq \limsup_{n \rightarrow \infty} \lambda(A_n).$$

**Problem 7:** Show there is no Lebesgue measurable set  $A \subseteq \mathbb{R}$  such that for every real interval  $(a, b)$  we have  $\lambda(A \cap (a, b)) = \frac{(b-a)}{2}$ .

## Part III (do any 3 problems)

**Problem 8:** Let  $E \subset \mathbb{R}$  have finite Lebesgue measure. Show that if  $f \in L^\infty(E)$ , then  $f \in L^p(E)$  for all  $p \geq 1$  and  $\|f\|_\infty = \lim_{p \rightarrow \infty} \|f\|_p$ .

**Problem 9:** Let  $f \in L^1(\mathbb{R})$  be nonnegative and define  $\mu(E) = \int_E f \, d\lambda$  for every Lebesgue measurable set  $E \subseteq \mathbb{R}$ . Show that  $\mu \ll \lambda$ .

**Problem 10:** Show that if  $f \in L^1([0, 2\pi])$ , then

$$\lim_{n \rightarrow \infty} \int_{[0, 2\pi]} f(x) \cos(nx) \, dx = 0.$$

**Problem 11:** Let  $0 < p < q < \infty$  and  $\lambda(E) < \infty$ . Show that  $L^q(E) \subseteq L^p(E)$ .



# ANALYSIS QUALIFYING EXAM

AUGUST 12, 2009

## Section 1

Determine whether the following are true or false and state your reasons. Each is worth 10 points: 5 for a correct answer and 5 for a correct reason.

**Problem 1.** Every bounded sequence in a Hilbert space has a convergent subsequence.

**Problem 2.** If  $f : [0, 1] \rightarrow \mathbb{R}$  has bounded variation, then there is a countable subset  $S \subset [0, 1]$  such that  $f'$  exists everywhere in  $[0, 1] \setminus S$ .

**Problem 3.** If  $\lambda$  is Lebesgue measure on  $\mathbb{R}$  and  $\#$  is counting measure on  $\mathbb{R}$ , then  $\lambda \ll \#$ .

**Problem 4.** If  $f_n$  is a sequence of real-valued functions on  $\mathbb{R}$  such that  $f_n$  converges to  $f$  pointwise, then  $f_n$  converges to  $f$  in measure.

**Problem 5.**  $L^1([0, 1]) \subset L^2([0, 1])$

The problems in the following two sections require proofs. Each problem is worth 25 points. Do five of them, with at least two from each section. At most five will be graded, so, if you submit more than five of them, indicate which five you want graded.

## Section 2

**Problem 6.** If  $A$  is an uncountable subset of  $\mathbb{R}^n$ , then  $A$  has a limit point.

**Problem 7.** If  $f \in L^1([0, 1])$  and  $\alpha \in (0, 1)$  such that  $\int_A f = 0$  whenever  $\lambda(A) = \alpha$ , then  $f = 0$  a.e. ( $\lambda$  is Lebesgue measure.)

**Problem 8.** Give an example of a function  $f : [0, 1] \rightarrow [0, 1]$  which is strictly increasing but is not absolutely continuous. Justify your answer.

**Problem 9.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an arbitrary function with

$$C(f) = \{x : f \text{ is continuous at } x\}.$$

Prove that  $C(f)$  is a  $G_\delta$  set. (A set is called a  $G_\delta$  set, if it can be written as the intersection of a countable collection of open sets.)

**Problem 10.** Assuming the statement of Problem 9, conclude there is no function  $f : \mathbb{R} \rightarrow \mathbb{R}$  with  $C(f) = \mathbb{Q}$ .

**Problem 11.** If  $f_n : [0, 1] \rightarrow \mathbb{R}$  is a sequence of measurable functions such that  $\sum_{n \in \mathbb{N}} \lambda(\{x : |f_n(x)| > 1/n\})$  converges, then  $f_n \rightarrow 0$  a.e. ( $\lambda$  denotes Lebesgue measure.)

**Problem 12.** Let  $(X, \rho)$  and  $(Y, \sigma)$  be metric spaces and  $F : X \rightarrow Y$  be surjective and continuous.

- (a) If  $X$  is separable, must  $Y$  be separable?
- (b) If  $X$  is complete must  $Y$  be complete?

### Section 3

**Problem 13.** Let  $(X, \mathcal{M}, \mu)$  be a measure space. If  $f : X \rightarrow \mathbb{R}$  is such that  $\{x : f(x) > r\} \in \mathcal{M}$  whenever  $r \in \mathbb{Q}$ , then must  $f$  be measurable?

**Problem 14.** If  $K$  is a compact subset of the metric space  $(M, d)$  and  $f : M \rightarrow (0, \infty)$  is continuous, then there is an  $\varepsilon > 0$  such that  $f(x) \geq \varepsilon$  for all  $x \in K$ .

**Problem 15.** Let  $\mu$  and  $\nu$  be finite measures on a measurable space  $(X, \mathcal{M})$ . Prove that if  $\mu \ll \nu$  and  $\nu \ll \mu$ , then

$$\frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\mu} = 1 \quad \mu\text{-a.e.}$$

**Problem 16.** Prove that a measure space  $(X, \mathcal{S}, \mu)$  is  $\sigma$ -finite if and only if there is a positive-valued  $f \in L^1(\mu)$ .

**Problem 17.** Let  $(X, \mathcal{S}, \mu)$  be a finite measure space. If  $f \in L^1(\mu)$ , then compute

$$\lim_{n \rightarrow \infty} \int_X |f(x)|^{1/n} d\mu(x).$$

# ANALYSIS QUALIFIER

MAY 7, 2007

## 1. TRUE/FALSE

Determine whether the following statements are true or false and explain how you reached your conclusion. Each problem is worth 10 points. You receive 5 points for a correct answer and 5 additional points for a correct reason.

**Problem 1.** If

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

then  $f$  is Riemann integrable over any compact interval.

**Problem 2.** If  $f$  is continuous on  $[0, 1]$  and  $f'(x) \leq -\frac{1}{2}$  a.e., then  $f(0) > f(1)$ .

**Problem 3.** If  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  is a sequence of functions converging pointwise to  $f$ , then  $f_n$  converges in measure to  $f$ .

**Problem 4.** Let  $S$  be a countable and dense subset of  $[a, b]$ . There is a function  $f : [a, b] \rightarrow \mathbb{R}$  that is of bounded variation, continuous at each point of  $S$  and discontinuous everywhere else.

**Problem 5.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and let  $a \in \mathbb{R}$ . If  $f'(a) > 0$  then  $f$  is increasing on a neighborhood of  $a$ .

## 2. PROOFS

Each of the following problems is worth 25 points. At most three of your solutions will be graded. If you submit more than three solutions, please indicate which three should be graded.

**Problem 1.** If  $E \subset \mathbb{R}$  is a Lebesgue measurable set such that  $|E \cap I| \leq \frac{2}{3}|I|$  for every bounded interval  $I$ , then  $|E| = 0$ .

**Problem 2.** Let  $f_n$  and  $f$  be functions from  $L^1([0, 1])$  such that  $f_n \rightarrow f$  in  $L^1([0, 1])$ . There is a subsequence  $f_{n_j}$  of  $f_n$  converging to  $f$  a.e. on  $[0, 1]$ .

**Problem 3.** Let  $\{r_k\}$  be an enumeration of the rational numbers in  $\mathbb{R}$ . Define

$$A_{j,k} = \left( r_k - 2^{-(j+k)}, r_k + 2^{-(j+k)} \right), \quad A_j = \bigcup_{k=1}^{\infty} A_{j,k} \quad \text{and} \quad A = \bigcap_{j=1}^{\infty} A_j.$$

Prove that  $\mathbb{Q} \subset A$  and  $|A| = 0$ . Is  $A = \mathbb{Q}$ ?

**Problem 4.** If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and  $F \subset \mathbb{R}$  is an  $\mathbf{F}_\sigma$  set, then  $f(F)$  is measurable.

**Problem 5.** Let  $K \subset \mathbb{R}$  be compact,  $x_0 \in K$  and  $x_n$  be a sequence in  $K$ . If every convergent subsequence of  $x_n$  converges to  $x_0$ , then  $x_n \rightarrow x_0$ .

## 3. MORE PROOFS

Each of the following problems is worth 25 points. At most three of your solutions will be graded. If you submit more than three solutions, please indicate which three should be graded.

**Problem 1.** Let  $\mu, \mu_n$  be finite measures defined on a measure space  $(X, \mathcal{M})$ . Suppose that

$$\lim_{n \rightarrow \infty} \int_E d\mu_n = \int_E d\mu$$

for every set  $E \in \mathcal{M}$ . Prove that

$$\lim_{n \rightarrow \infty} \int_X f d\mu_n = \int_X f d\mu$$

for every bounded measurable function  $f : X \rightarrow \mathbb{R}$ .

**Problem 2.** Give an example of a signed measure  $\nu$  defined on  $\mathbb{R}$  with the property that both sets in the Hahn decomposition are dense in  $\mathbb{R}$ .

**Problem 3.** Let  $(X, \mathcal{M}, \mu)$  be a measure space and  $f_n : X \rightarrow \mathbb{R}$  be a sequence of measurable functions. The set  $C = \{x : f_n(x) \text{ converges}\} \in \mathcal{M}$ .

**Problem 4.** Let  $1 \leq p < \infty$ . Then  $L^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$  is dense in  $L^p(\mathbb{R})$ .

**Problem 5.** If  $f \in L^1(\mathbb{R})$ , then  $\lim_{n \rightarrow \infty} \frac{1}{2n} \int_{-n}^n f = 0$ .

# ANALYSIS QUALIFIER

OCTOBER 10, 2007

## 1. TRUE/FALSE

Determine whether the following statements are true or false and explain how you reached your conclusion. Each problem is worth 10 points. You receive 5 points for a correct answer and 5 additional points for a correct reason.

**Problem 1.** Every finite subset of a metric space is nowhere dense.

**Problem 2.** There is a set  $S \subset \mathbb{Q}$  such that  $S$  is not a Borel set.

**Problem 3.** If  $f : [0, 1] \rightarrow \mathbb{R}$  is injective, then there must be a subinterval of  $[0, 1]$ , on which  $f$  is monotone.

**Problem 4.** Given a differentiable function  $f : [0, \infty) \rightarrow \mathbb{R}$ , assume that

$$\lim_{x \rightarrow \infty} f(x) = 0,$$

then

$$\lim_{x \rightarrow \infty} f'(x) = 0.$$

**Problem 5.**  $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$

## 2. PROOFS

Each of the following problems is worth 25 points. At most three of your solutions will be graded. If you submit more than three solutions, please indicate which three should be graded.

**Problem 1.** If  $f : [a, b] \rightarrow \mathbb{R}$  is a function of bounded variation and there is an  $\alpha > 0$  such that  $|f(x)| \geq \alpha$  on  $[a, b]$ , then  $1/f$  is of bounded variation on  $[a, b]$ .

**Problem 2.** Let  $\lambda$  be Lebesgue measure on  $\mathbb{R}$ . Show there does not exist a measurable set  $A \subset \mathbb{R}$  such that for every interval  $(a, b)$ ,  $\lambda(A \cap (a, b)) = (b - a)/2$ .

**Problem 3.** Prove  $L^2([0, 1]) \subset L^1([0, 1])$ .

**Problem 4.** Give an example of a set  $S \subset \mathbb{R}$  which is both nowhere dense and of positive measure. (Include details of the construction.)

**Problem 5.** State the Monotone Convergence Theorem and use it to prove Fatou's Lemma.

## 3. MORE PROOFS

Each of the following problems is worth 25 points. At most three of your solutions will be graded. If you submit more than three solutions, please indicate which three should be graded.

**Problem 1.** If  $(X, \Sigma, \mu)$  is a measure space and  $f : X \rightarrow (0, \infty)$  is a measurable function, then  $1/f$  is measurable.

**Problem 2.** Let  $\lambda$  be Lebesgue measure on  $\mathbb{R}$  and  $\varepsilon > 0$ . If  $S \subset \mathbb{R}$  is bounded and measurable, then  $S = K \cup T$  such that  $K$  is compact and  $\lambda(T) < \varepsilon$ .

**Problem 3.** If  $f \in L^1([0, 1])$ , then

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) \sin nx \, dx = 0.$$

**Problem 4.** If  $\lambda$  is Lebesgue measure on  $\mathbb{R}$  and a measure  $\mu$  is defined by  $\mu(A) = \lambda(A \cap [-1, 1])$ , then prove that  $\mu \ll \lambda$  and explicitly find the Radon-Nikodym derivative  $d\mu/d\lambda$ .

**Problem 5.** Let  $1 \leq p < \infty$ . Every Cauchy sequence in  $L^\infty([0, 1])$  is a Cauchy sequence in  $L^p([0, 1])$ .

5/22/06      **Spring Analysis Qualifier**

Name:

The test consists of three parts.

1. The first part consists of 5 True/False questions. This part is **mandatory**.
2. You will be awarded credit for 3 out of 5 questions in the second part, and 3 out of 5 questions in the third part. In the following boxes, *please circle the 3* that you would like us to grade.

Please do **all** of the following:

<i>Part I</i>	(a)	(b)	(c)	(d)	(e)	<i>Total</i>
<i>Max. pts</i>	5	5	5	5	5	25
<i>Grade</i>						

Please **circle exactly 3** of the following:

<i>Part II</i>	1	2	3	4	5	<i>Total</i>
<i>Max. pts</i>	25	25	25	25	25	75
<i>Grade</i>						

Please **circle exactly 3** of the following:

<i>Part II</i>	1	2	3	4	5	<i>Total</i>
<i>Max. pts</i>	25	25	25	25	25	75
<i>Grade</i>						

**Note:** Please note that a complete and correct solution will carry far more weight than several sparsely supported "solution sketches".

**FINAL SCORE (out of 175):**

**PASS**

**FAIL**

**PART I.** Mandatory. State whether each of the following statements is True (T) or False (F). Support your assertion with a proper justification. You will receive 2 points for the correct choice and 3 points for the justification.

(a) Given a fat Cantor set  $F \subseteq [0, 1]$ , there is a function of bounded variation on  $[0, 1]$  which is non-differentiable precisely on  $F$ .

T F

(b) If  $C_k \subseteq \mathbf{R}$  is a nested sequence of non-empty, closed sets, then  $\bigcap C_k \neq \emptyset$

T F

(c) For any function  $f : [0, 1] \rightarrow [0, 1]$ , if  $f' = 0$  a.e., then  $f$  is a constant.

T F

(d) Let  $f_n, f$  be finite, real valued measurable functions on  $[0, 1]$  equipped with Lebesgue measure. Then  $f_n \rightarrow f$  in  $L^1[0, 1]$  implies  $f_n \rightarrow f$  a.e.

T F

(e) Suppose  $f : \mathbf{R} \rightarrow \mathbf{R}$  and  $f(x) = x_0$  for all  $x \in \mathbf{Q}$ . Then  $f$  is a constant.

T F



**PART II.** Please choose 3 out of 5 from the following:

1. Define an  $F_\sigma$ -set. If  $X$  is a topological space and  $f : X \rightarrow \mathbf{R}$ , show that the set of points at which  $f$  is not continuous is an  $F_\sigma$ -set.
2. For a function  $f : [0, 1] \rightarrow \mathbf{R}$  define what it means for  $f$  to be of bounded variation. Let  $f(0) = 0$  and  $f(x) = x \sin(1/x)$ ,  $x \neq 0$ . Show that  $f$  is not of bounded variation.
3. Define a separable metric space. Prove that every compact metric space is separable.
4. State the Baire Category Theorem. Use this to prove that  $\mathbf{R}$  is uncountable.
5. Define a Hilbert space. For a Hilbert space  $\mathcal{H}$  and an orthonormal basis  $B$  of  $\mathcal{H}$ , prove that  $\|x\|^2 = \sum_{x_j \in B} \langle x, x_j \rangle^2$ ,  $\forall x \in \mathcal{H}$ .

**PART III.** Please choose 3 out of 5 from the following:

1. Suppose  $A \subseteq \mathbf{R}$  is an uncountable set. Then  $A$  has uncountably many accumulation points.
2. Let  $f$  be a real-valued function on  $[0, 1]$ , and let  $\lambda^*$  denote the Lebesgue outer measure. Suppose  $E \subseteq [0, 1]$  and  $f'$  exists and is bounded on  $E$  by a constant bound  $M$ . Prove that  $\lambda^*(f(E)) \leq M\lambda^*(E)$ .
3. Suppose  $\nu, \mu$  are  $\sigma$ -finite measures on the measurable space  $(X, \mathcal{M})$ , such that  $\nu \ll \mu$ . Prove that if  $g \in L^1(\nu)$ , then  $g(d\nu/d\mu) \in L^1(\mu)$  and  $\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$ .
4. Let  $1 < p < \infty$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . Prove that for all  $f \in (l^p)^*$  there is a fixed  $t \in l^q$  such that  $f(s) = \sum s_n t_n$  for all  $s \in l^p$ .
5. Let  $\mathcal{P}$  denote the space of all polynomials on  $[0, 1]$  with  $L^\infty[0, 1]$  norm. If  $F : \mathcal{P} \rightarrow \mathcal{P}$  is defined by

$$F\left(\sum_{k=0}^n a_k x^k\right) = \sum_{k=0}^n a_k x^{k+1},$$

show that  $F$  is continuous and find  $\|F\|$ .

## 10/24/2006 Analysis Qualifier

The test consist of 3 parts.

Do all five of T/F. They are worth 10 pts each. 3 points for getting T/F correct and 7 points for a correct explanation.

Do 3 of five from section 2. Clearly, state which problems you want graded. Each problem is worth 25 points.

Do 3 of five from section 3. Clearly state which problems you want graded. Each problem is worth 25 points.

**Note: A complete and correct solution will carry far more weight than several sparsely supported "solution sketches".**

**Part I.** Decide if the following statements are True or False. You will receive 2 points for deciding if the statement is T/F and 3 points for a correct explanation.

1. Suppose  $f_n : [0, 1] \rightarrow \mathbb{R}$  and  $\{f_n\}$  converges uniformly to the zero function. Then, some  $f_n$  must be Riemann integrable.
2. There are disjoint sets  $A, B \subseteq \mathbb{R}$  both of which are dense in  $\mathbb{R}$  and both of them have positive measure.
3. Suppose  $f : [0, 1] \rightarrow \mathbb{R}$  is such that  $|f(x) - f(y)| \leq |x - y|$  for all  $x, y \in [0, 1]$ . Then,  $f$  is differentiable a.e.
4. Every infinite bounded subset of  $C([0, 1])$  (the space of continuous functions on  $[0, 1]$  endowed with the sup norm) has a limit-point.
5. If  $f : [0, 1] \rightarrow \mathbb{R}$  is continuous and of bounded variation and  $f'(x) = 0$  for a.e.  $x \in [0, 1]$ , then  $f$  is constant.

**Part II.** Each of the following problem is worth 25 points. Do 3 of the following. Clearly state which problems you would like graded.

1. State the definition of compactness. Prove that  $M \subseteq \mathbb{R}$  is compact iff every infinite subset of  $M$  has a limit-point in  $M$ .
2. State a version of the Baire category theorem. Use it to show that the set of irrational numbers is not  $F_\sigma$ .
3. State a necessary and sufficient condition for a function  $f : [0, 1] \rightarrow \mathbb{R}$  to be Riemann integrable. Give an example of a closed set whose characteristic function is not Riemann integrable. Explain why your example works.
4. Prove that the set of Borel sets is the smallest  $\sigma$ -algebra containing all sets of the form  $[a, \infty)$ .
5. Suppose  $f : [0, \infty) \rightarrow \mathbb{R}$  is twice differentiable and  $\lim_{x \rightarrow \infty} f''(x) = 0$ . Determine

$$\lim_{x \rightarrow \infty} [f(x - 5) + f(x + 5) - 2f(x)].$$

Give a complete justification for your answer, clearly stating which theorems you are using.

**Part III.** Each of the following problem is worth 25 points. Do 3 of the following. Clearly state which problems you would like graded.

1. Let  $A \subseteq \mathbb{R}$  be such that  $\lambda(A) > 0$ . ( $\lambda$  denotes the Lebesgue measure.) Show that there is a closed interval  $I$  such that  $9\lambda(A \cap I) > \lambda(I)$ .
2. Suppose  $f : [0, 1] \rightarrow \mathbb{R}$  is an  $L^1$  function. For a Borel set  $B \subseteq \mathbb{R}$ , define

$$\mu(B) = \int_{[0,1]} f d\lambda.$$

( $\lambda$  denotes the Lebesgue measure.) Explain why  $\mu$  is absolutely continuous with respect to  $\lambda$ . Give the Hahn-Decomposition of  $\mu$ .

3. State the Lebesgue dominated convergence theorem. Use it prove that if  $\{f_n\}$  is a sequence of Lebesgue integrable functions defined on  $[0, 1]$  which converges uniformly to  $f$ , then  $f$  is Lebesgue integrable and  $\lim_{n \rightarrow \infty} \int_0^1 f_n = \int_0^1 f$ .
4. For each  $n \in \mathbb{N}$ , define an operator  $T_n$  on  $C([0, 1])$ , the space of continuous functions endowed with the sup norm, as follows.

$$T_n(f) = \frac{1}{n} \sum_{i=0}^{n-1} f(i).$$

Prove that each  $T_n$  is a continuous linear operator on  $C([0, 1])$ . Find an operator  $T$  so that  $\{T_n\}$  converges to  $T$ .

5. Prove that  $L^\infty([0, 1])$  has an uncountable set which has no accumulation point and conclude that it is not separable.

5/13/05

**Spring Analysis Qualifier**

Name:

The test consists of three parts.

1. The first part consists of 5 True/False questions. This part is mandatory.
2. You will be awarded credit for 3 out of 5 questions in the second part, and 3 out of 5 questions in the third part. In the following boxes, *please circle the 3* that you would like us to grade.

Please do **all** of the following:

<i>Part I</i>	(a)	(b)	(c)	(d)	(e)	<i>Total</i>
<i>Max. pts</i>	5	5	5	5	5	25
<i>Grade</i>						

Please **circle exactly 3** of the following:

<i>Part II</i>	1	2	3	4	5	<i>Total</i>
<i>Max. pts</i>	25	25	25	25	25	75
<i>Grade</i>						

Please **circle exactly 3** of the following:

<i>Part II</i>	1	2	3	4	5	<i>Total</i>
<i>Max. pts</i>	25	25	25	25	25	75
<i>Grade</i>						

**Note:** Please note that a complete and correct solution will carry far more weight than several sparsely supported "solution sketches".

**FINAL SCORE (out of 175):**

**PASS**

**FAIL**

**PART I.** State whether each of the following statements is True (T) or False (F). Support your assertion with a proper justification. You will receive 2 points for the correct choice and 3 points for the justification.

(a) There exists a monotone function  $f : [0, 1] \rightarrow [0, 1]$  such that  $f$  is discontinuous precisely at points in the Cantor ternary set.

T      F

(b) Let  $\{f_n\}$  be a sequence of non-Riemann integrable functions on  $[0, 1]$ . If  $f_n \rightarrow f$  uniformly on  $[0, 1]$ , then  $f$  is also not Riemann integrable.

T      F

(c) There is a dense open subset  $A$  of the reals, such that  $A$  has an uncountable complement.

T      F

(d) There exists a monotone function  $f : [0, 1] \rightarrow [0, 1]$  such that  $f$  is non-differentiable precisely at points in a Cantor set of positive Lebesgue measure.

T      F

(e) Every closed and bounded subset of a complete metric space is compact.

T      F



**PART II.** 1. Clearly state the Cauchy-Schwarz inequality. Show that if  $\sum_{n=1}^{\infty} a_n^2$  converges absolutely, then  $\sum_{n=1}^{\infty} \frac{a_n}{n}$  must also converge absolutely.

2. Clearly state the Heine-Borel Theorem. For a bounded open set  $A$  of real numbers, give an explicit construction of an open cover that has no finite subcover.

3. Clearly state Fatou's lemma and the Dominated Convergence Theorem. Now use Fatou's lemma to prove the Dominated Convergence Theorem.

4. Clearly state the Riesz Representation Theorem for  $\mathcal{L}^p[0, 1]$  for  $1 < p < \infty$ . For  $g \in \mathcal{L}^1[0, 1]$ , prove that  $T(f) = \int_0^1 fg \, dx$  defines a bounded linear functional on  $\mathcal{L}^\infty[0, 1]$  and find  $\|T\|$ .

5. For a continuous function  $f$  on  $[0, 1]$ , clearly define its Riemann integral as a limit of partial sums using uniform partitions. Show that

$$\lim_{n \rightarrow \infty} \left( \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n} \right) = \ln 2.$$

**PART III.** 1. Suppose  $f \in \mathcal{L}^1[0, 1]$  and set

$$F(x) = \int_0^x f(t)dt.$$

Prove that  $F$  is of bounded variation on  $[0, 1]$ .

2. Let  $f : \mathbf{R} \rightarrow \mathbf{R}$ . Show that  $f$  is discontinuous on a set of first category in  $\mathbf{R}$  if and only if  $f$  is continuous at a dense set of points.

3. Let  $\mathcal{H}$  be a separable Hilbert space, and let  $\{\phi_k\}$  be an orthonormal set in  $\mathcal{H}$ . Show that for any  $x \in \mathcal{H}$ , we have

$$\sum_{k=1}^{\infty} |\langle x, \phi_k \rangle|^2 \leq \|x\|^2.$$

State a sufficient condition for equality.

4. Suppose that  $\nu$  is a  $\sigma$ -finite signed measure and  $\mu$  is a  $\sigma$ -finite measure on  $(X, \mathcal{M})$  such that  $\nu \ll \mu$ . Show that if  $g \in \mathcal{L}^1(\nu)$ , then

$$g \frac{d\nu}{d\mu} \in \mathcal{L}^1(\mu) \text{ and } \int g d\nu = \int g \frac{d\nu}{d\mu} d\mu,$$

where  $\frac{d\nu}{d\mu}$  is the Radon-Nikodym derivative of  $\nu$  with respect to  $\mu$ .

5. Let  $C$  denote the Cantor ternary set in  $[0, 1]$ . Show that

$$C - C = \{a - b \mid a, b \in C\} = [-1, 1].$$

10/26/05      **Fall Analysis Qualifier**

Name:

The test consists of three parts.

1. The first part consists of 5 True/False questions. This part is **mandatory**.
2. You will be awarded credit for 3 out of 5 questions in the second part, and 3 out of 5 questions in the third part. In the following boxes, *please circle the 3* that you would like us to grade.

Please do **all** of the following:

<i>Part I</i>	(a)	(b)	(c)	(d)	(e)	<i>Total</i>
<i>Max. pts</i>	5	5	5	5	5	25
<i>Grade</i>						

Please **circle exactly 3** of the following:

<i>Part II</i>	1	2	3	4	5	<i>Total</i>
<i>Max. pts</i>	25	25	25	25	25	75
<i>Grade</i>						

Please **circle exactly 3** of the following:

<i>Part II</i>	1	2	3	4	5	<i>Total</i>
<i>Max. pts</i>	25	25	25	25	25	75
<i>Grade</i>						

**Note:** Please note that a complete and correct solution will carry far more weight than several sparsely supported "solution sketches".

**FINAL SCORE (out of 175):**

**PASS**

**FAIL**

**PART I.** State whether each of the following statements is True (T) or False (F). Support your assertion with a proper justification. You will receive 2 points for the correct choice and 3 points for the justification.

(a) Let  $\{f_n\}$  be a sequence of Riemann integrable functions on  $[0, 1]$ . If  $f_n \rightarrow f$  uniformly on  $[0, 1]$ , then  $f$  is also Riemann integrable. T      F

(b) There is a monotone increasing function on  $[0, 1]$  that is not absolutely continuous. T      F

(c) Every non-empty perfect nowhere dense subset of the reals must have Lebesgue measure zero. T      F

(d) Let  $f_n, f$  be finite, real valued measurable functions on  $[0, 1]$  equipped with Lebesgue measure. Then  $f_n \rightarrow f$  in measure implies  $f_n \rightarrow f$  a.e. T      F

(e) There is a function of bounded variation on  $[0, 1]$  which is non-differentiable precisely on the Cantor ternary set. T      F

- PART II.**
1. Clearly state the Axiom of Choice. Show that the Axiom of Choice follows from the Well Ordering Principle (every set can be well-ordered).
  2. Clearly state the Baire Category Theorem. Show that a dense  $\mathcal{G}_\delta$  set in a complete metric space is residual, i.e. it must be the complement of a first-category set.
  3. Clearly define an absolutely continuous function on  $[a, b]$ . Show that if  $f$  is absolutely continuous on  $[a, b]$  then for every  $E \subseteq [a, b]$ , of Lebesgue measure zero (i.e.  $\lambda(E) = 0$ ), we must have  $\lambda(f(E)) = 0$ .
  4. Clearly define a separable metric space. Let  $\mathcal{C}[a, b]$  denote the space of continuous functions on  $[a, b]$ . Is  $\mathcal{C}[a, b]$  separable? Sketch a proof.
  5. Let  $X, Y$  be normed linear spaces and let  $T : X \rightarrow Y$ . Clearly define what it means for  $T$  to be a bounded linear operator. Prove that a bounded linear operator is continuous.

**PART III.** 1. Suppose  $f : \mathbf{R} \rightarrow \mathbf{R}$ . Then there is a constant  $M$  such that  $|f(x) - f(y)| \leq M|x - y|$  for all  $x, y \in \mathbf{R}$  iff  $f$  is absolutely continuous and  $|f'| \leq M$  Lebesgue a.e.

2. Let  $(X, \mathcal{B}, \mu)$  be a measure space,  $f \in \mathcal{L}^1(\mu)$ . Define

$$\nu(A) = \int_A f d\mu, \quad A \in \mathcal{B}$$

Show that  $\nu$  is a signed measure on  $X$  and find a Hahn decomposition for  $\nu$ .

3. Let  $(X, \rho)$  be a complete metric space and suppose  $f : X \rightarrow X$  has the property that

$$\rho(f(x), f(y)) < C\rho(x, y)$$

for some  $0 < C < 1$  and for all  $x, y \in X$ ,  $x \neq y$ . Show that  $f$  has a unique fixed point.

4. A sequence  $(f_n)$  in  $\mathcal{L}^p([0, 1])$  is said to converge weakly to a function  $f \in \mathcal{L}^p([0, 1])$  if  $\int f_n g \rightarrow \int f g$  for all  $g \in \mathcal{L}^q([0, 1])$ , where  $p, q$  are conjugates. Show that every orthonormal sequence converges weakly to 0 in  $\mathcal{L}^2([0, 1])$ .

5. Let  $\mathcal{C}[0, 1]$  denote the space of continuous functions on  $[0, 1]$ . For a fixed choice of  $t_1, t_2, \dots, t_n \in [0, 1]$  and  $\lambda_1, \dots, \lambda_n \in \mathbf{R}$ , we may define  $F : \mathcal{C}[0, 1] \rightarrow \mathbf{R}$  as

$$F(f) = \sum_{i=1}^n \lambda_i f(t_i).$$

Show that  $F$  is a linear functional on  $\mathcal{C}[0, 1]$  and find the norm of  $F$ .

## 1 Basic Analysis.

Do **three** problems from this section. Clearly state which three problems you would like graded.

1. 1a. State some version of the Baire Category Theorem.
  - 1b. Prove that the set of irrational numbers is not the countable union of closed subsets of  $\mathbb{R}$ .
2. Suppose that  $(X, d)$  is a separable metric space. Show that every uncountable subset of  $X$  has a limit-point in  $X$ .
3. 3a. State a condition which is necessary and sufficient for a function  $f : [0, 1] \rightarrow \mathbb{R}$  to be Riemann integrable.
  - 3b. Give an example of a characteristic function of a closed set which is not Riemann integrable. Explain why your example works.
4. Consider the power series:

$$f(x) = \sum_{n=0}^{\infty} \frac{n^2}{3^n} x^n.$$

Show that  $f$  is continuous and differentiable on  $(-3, 3)$ .

## 2 Measure and Integration.

Do **three** problems from this section. Clearly state which three problems you would like graded.

5. 5a. State what it means for a function  $f : [0, 1] \rightarrow \mathbb{R}$  to be absolutely continuous.
  - 5b. Give an example of a continuous nondecreasing function  $f : [0, 1] \rightarrow [0, 1]$  such that  $f$  is not absolutely continuous. Explain why your function is not absolutely continuous.
6. Suppose that  $M \subset [0, 1]$  is such that  $M \cap P \neq \emptyset$  and  $M^c \cap P \neq \emptyset$  for every closed set  $P \subseteq [0, 1]$  of positive measure. Show that  $M$  is non-measurable.
7. Suppose that  $f \in L^1([0, 1])$  and  $\{g_n\}$  is a sequence of bounded measurable functions defined on  $[0, 1]$  which converges uniformly to  $g$ . Show that

$$\lim_{n \rightarrow \infty} \int_{[0,1]} f g_n = \int_{[0,1]} f g.$$

8. Show that the smallest  $\sigma$ -algebra containing  $\mathcal{G} = \{[a, b] : a, b \in \mathbb{R}\}$  is the set of Borel sets.
9. Suppose  $f, g \in L^1((-\infty, \infty))$ . Define  $h(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt$  for all  $x \in \mathbb{R}$ . Show that  $\|h\|_1 = \|f\|_1 \|g\|_1$  and conclude that  $h$  is finite a.e.

### 3 Functions Spaces.

Do **two** problems from this section. Clearly state which two problems you would like graded.

10. Define  $T : C([0, 1]) \rightarrow \mathbb{R}$  by  $T(f) = \sum_{n=0}^{\infty} \frac{1}{2^n} f(\frac{1}{2^n})$ .
  - 10a. Explain why  $T(f)$  is finite.
  - 10b. Show that  $T$  is a bounded linear operator on  $C([0, 1])$ .
  - 10c. Find a BV function  $g : [0, 1] \rightarrow \mathbb{R}$  such that  $T(f) = \int_0^1 f dg$  for all  $f \in C([0, 1])$ .
11. Suppose that  $\{f_n\}$  is a sequence of differentiable functions defined on  $[0, 1]$  such that  $f_n(0) = 0$  for all  $n \in \mathbb{N}$  and  $|f'_n(x)| \leq 1$  for all  $x \in [0, 1]$  and for all  $n \in \mathbb{N}$ . Show that  $\{f_n\}$  has a subsequence which converges uniformly to some function  $f$  which is Lipschitz.
12. Find an uncountable subset  $\mathcal{U}$  of  $L^\infty([0, 1])$  such that  $\|f - g\|_\infty = 2$  for all  $f, g \in \mathcal{U}$  with  $f \neq g$ .



# Analysis Qualifier Exam

October 13, 2004

Following are ten problems divided into two groups of five. Do five problems, including at most three from each group.

## Group 1

**Problem 1.** Find a set  $S \subset \mathbb{R}$  of Lebesgue measure zero that is uncountable in every interval.

**Problem 2.** If  $f$  is Lebesgue integrable on  $[0, 1]$ , then

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) \cos(nx) dx = 0.$$

**Problem 3.** Suppose that  $\lambda$  is Lebesgue measure and  $\mu$  is counting measure on  $I = [0, 1]$ . (Regard both as Borel measures.) If  $\Delta = \{(x, x) : x \in I\}$  and  $f$  is the characteristic function of  $\Delta$ , then calculate

$$\int_I \left( \int_I f d\lambda \right) d\mu, \int_I \left( \int_I f d\mu \right) d\lambda \text{ and } \int_{I \times I} f d\mu \times d\lambda.$$

Reconcile your results with Fubini's theorem.

**Problem 4.** Suppose  $f \in L^1([0, 1])$  and

$$\left| \int_a^b f \right| \leq b - a$$

whenever  $0 \leq a < b \leq 1$ . Prove  $|f(x)| \leq 1$  a.e.

**Problem 5.** Let

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

(a) Show that the Lebesgue integral  $\int_0^\infty f$  does not exist.

(b) Show that the improper Riemann integral  $\int_0^\infty f(x) dx$  does exist.

## Group 2

**Problem 6.** If  $f$  is monotone on  $[0, 1]$ , then there is a sequence of continuous functions  $f_n \rightarrow f$ .

**Problem 7.** If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then

$$G = \{(x, f(x)) : a \leq x \leq b\} \subset \mathbb{R}^2$$

has Lebesgue measure 0.

**Problem 8.** Prove that a dense  $G_\delta$  subset of  $\mathbb{R}$  is uncountable.

**Problem 9.** If  $f \in L^\infty([0, 1])$ , then  $f \in L^p([0, 1])$  for all  $p \geq 1$  and

$$\|f\|_\infty = \lim_{p \rightarrow \infty} \|f\|_p.$$

**Problem 10.** If  $f_n : [a, b] \rightarrow \mathbb{R}$  is a sequence of continuous functions that converges uniformly, then

$$\mathcal{F} = \{f_n : n = 1, 2, 3, \dots\}$$

is equicontinuous.