

## Complex Analysis Qualifying Examination

1. **THE COMPLEX PLANE AND ELEMENTARY FUNCTIONS:** Complex numbers; stereographic projection; simple and multiple connectivity, elementary functions.
2. **ANALYTIC FUNCTIONS:** Cauchy–Riemann equations; power series; the zeros of an analytic function;
3. **COMPLEX INTEGRATION:** Harmonic conjugate, mean value property, maximum principle, Cauchy's theorem; Cauchy's integral formula; residue theorem; computation of definite integrals by residues, Liouville's theorem, Morera's theorem;
4. **CONFORMAL MAPPING:** Linear fractional transformations; mappings by elementary functions; Riemann mapping theorem.
5. **SINGULARITIES:** Laurent series; classification of isolated singularities;
6. **LOGRITHMIC INTEGRAL:** The argument principle, Rouche's theorem, Hurwitz's theorem, winding numbers; open mapping theorem; Schwarz lemma; conformal self-maps of the unit disk;
7. **ANALYTIC CONTINUATION:** Schwarz reflection principle; analytic continuation along a path; harmonic functions, the Poisson integral formula;

### References:

- [1] T. W. Gamelin, *Complex analysis*, Springer, New York, 2001.
- [2] Lars V. Ahlfors, *Complex Analysis, 3rd Ed.*
- [3] John Mathews and Russell Howell, *Complex Analysis for Mathematics and Engineering*, (This is the textbook of Math 511. This book may provide introductory materials and exercises. )

# Complex Analysis Qualifier Exam

(Sample)

**PART I: DO ANY 6 PROBLEMS. (12 points each)**

1. Sketch the image of the sector  $\{0 < \arg z < \pi/6\}$  under the map  $w = z^i$ . Use only the principal branch. (Show your analysis.)
2. (a) Evaluate  $\int_{|z-1|=3} \frac{dz}{z(z^2-4)e^z}$ .  
 (b) Show that  $\int_{C_2^+(0)} \frac{\exp z^2}{(z-i)^4} dz = \frac{-4\pi}{3e}$ .
3. Show that if  $D$  is a bounded domain in the complex plane with piecewise smooth boundary, and if  $g(z)$  is a smooth function on  $D \cup \partial D$ , then  $\int_{\partial D} g(z) dz = 2i \iint_D \frac{\partial g}{\partial \bar{z}} dx dy$ .
4. (a) Find the Taylor series of  $f(z) = \frac{1-z}{z-4}$  for  $|z-3| < 1$ . ( $z_0 = 3$ )  
 (b) Find the Laurent series of  $f(z) = \frac{1-z}{z-4}$  for  $|z-3| > 1$ . ( $z_0 = 3$ )
5. (a) Find the isolated singularities of the following functions, and determine whether they are removable, essential or poles. Determine the order of any pole, and find principal part at each pole.

$$\frac{ze^z}{z^2 - 1}$$

(b) Find the zeros and orders of the function

$$\frac{\cos z - 1}{z^2}$$

6. (a) Find  $\text{Res}[f, 0]$  for

$$f(z) = z^{-3} \cos z$$

(b) Use Cauchy's residue theorem to find

$$\int_{C_{1.5}^+(0)} \frac{1}{z^4 + z^3 - 2z^2} dz$$

7. Use complex contour integral to evaluate

$$\int_0^{2\pi} \frac{1}{4 \sin \theta + 5} d\theta$$

8. Use the residue theory to show that

$$\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} = \frac{\pi}{\sqrt{2}}$$

9. Find the number of zeros of the polynomial  $p(z) = z^4 + z^3 + 4z^2 + 3z + 2$  in each quadrant.
10. How many roots does  $z^9 + z^5 - 8z^3 + 2z + 1$  have between the circles  $\{|z| = 1\}$  and  $\{|z| = 2\}$ .

11. Find the critical points and critical values of  $f(z) = z^2 + 1$ . Sketch the set of points  $z$  such that  $|f(z)| \leq 1$ , and locate the critical points of  $f(z)$  on the sketch.
12. Find the Cauchy integrals of  $f(z)$  along the unit circle  $\Gamma = \{|z| = 1\}$ , positively oriented,

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - \zeta} dz$$

for (a)  $f(z) = \bar{z}$ ; (b)  $f(z) = \Re\{z\}$

**PART II: DO ANY 2 PROBLEMS. (14 points each)**

1. (a) State the Rouché's theorem;  
 (b) Based on Rouché's theorem, prove the Hurwitz's theorem: Suppose  $\{f_k(z)\}$  is a sequence of analytic functions on a domain  $D$  that converges normally on  $D$  to  $f(z)$ , and suppose that  $f(z)$  has a zero of order  $N$  at  $z_0$ . Then there exists  $\rho > 0$  such that for  $k$  large,  $f_k(z)$  has exactly  $N$  zeros in the disk  $\{|z - z_0| < \rho\}$ , counting multiplicity, and these zeros converge to  $z_0$  as  $k \rightarrow \infty$ .
2. Prove the Open Mapping Theorem for Analytic Functions: If  $f(z)$  is analytic on a domain  $D$ , and  $f(z)$  is not constant, then  $f(z)$  maps open sets to open sets, that is,  $f(U)$  is open for each open subset  $U$  of  $D$ .
3. (1) State Schwartz's Lemma.  
 (2) Based on Schwartz's Lemma, prove that the conformal self-maps of the open unit disk  $D$  are precisely the fractional linear transformations of the form

$$f(z) = e^{i\phi} \frac{z - a}{1 - \bar{a}z}, \quad |z| < 1,$$

where  $a$  is complex,  $|a| < 1$ , and  $0 \leq \phi \leq 2\pi$ .

4. Prove the complex version of the Strict Maximum Principle: Let  $h$  be a bounded complex-valued harmonic function on a domain  $D$ . If  $h(z) \leq M$  for all  $z \in D$ , and  $h(z_0) = M$  for some  $z_0 \in D$ , then  $h(z)$  is constant on  $D$ .

# Complex Analysis Qualifier, August 2013

This test is divided into two sections of five problems each. Your solutions for three of the problems from each section will be graded. If you submit more than three solutions from a section, indicate the three solutions that should be graded.

$\Delta = \{z \in \mathbb{C} : |z| < 1\}$  is the unit disk.

## Section 1

**Problem 1.** Find a conformal map from  $G = \{z : |z| > 0 \text{ and } \frac{\pi}{4} < \arg z < \frac{\pi}{2}\}$  onto  $\Delta$ .

**Problem 2.** If  $u(x, y) = \sinh x \sin y$ , then find its harmonic conjugate  $v(x, y)$ .

**Problem 3.** Evaluate  $\int_0^\infty \frac{\cos ax}{x^2 + 1} dx$  for  $a > 0$ .

**Problem 4.** Evaluate  $\int_0^{2\pi} \frac{d\theta}{4 \sin \theta + 5}$ .

**Problem 5.** How many zeros does  $f(z) = z^9 - 5z^5 + 2z - 1$  have in  $\Delta$ ?

## Section 2

**Problem 6.** Determine the order and location of the zeros and poles for  $f(z) = \frac{\tan z}{z}$ .

**Problem 7.** Let  $f$  be entire and not constant. If  $w_0 \in \mathbb{C}$  and  $\varepsilon > 0$ , then there is a  $z_0 \in \mathbb{C}$  such that  $|f(z_0) - w_0| < \varepsilon$ .

**Problem 8.** If  $f$  has a pole of order  $m$  at  $z_0$  and  $P$  is a polynomial of degree  $n$ , then prove  $P \circ f$  has a pole of order  $mn$  at  $z_0$ .

**Problem 9.** Let  $\Omega$  be a simply connected domain,  $a \in \Omega$  and  $f : \Omega \rightarrow \Omega$  be analytic. If  $f(a) = a$  and  $|f'(a)| = 1$ , then prove  $f$  is a homeomorphism.

**Problem 10.** Suppose  $f$  is analytic on  $\Delta$ , continuous on  $\overline{\Delta}$  and real-valued on the unit circle. If  $f(0) = 1$ , determine  $f$ .

# Spring 2014 Complex Analysis Qualifying Exam

This test is divided into two sections of five problems each. Your solutions for three of the problems from each section will be graded. If you submit more than three solutions from a section, indicate which three solutions you would like to have graded.

$\Delta = \{z \in \mathbf{C} : |z| < 1\}$  is the unit disk in  $\mathbf{C}$ .

$i$  is the imaginary unit.

## Section 1

1. Evaluate  $\int_0^{2\pi} \frac{d\theta}{3 \cos \theta + 5}$
2. If  $u(x, y) = \cosh x \sin y$  then determine its harmonic conjugate  $v(x, y)$ .
3. How many zeros (including multiplicity) does  $f(z) = z^{10} + 6z^6 + z^3 + 2$  have in  $\Delta$ ?
4. Find a conformal map from  $D = \{z = x + iy : x < 0, y < 0\}$  onto  $\Delta$ .
5. Evaluate  $\int_0^\infty \frac{\cos x}{(x^2 + 1)^2} dx$ .

## Section 2

1. Suppose that  $f$  is continuous on  $\overline{\Delta}$  and analytic in  $\Delta$ . Prove that if  $f(z)$  is real-valued whenever  $|z| = 1$ , then  $f$  is constant.
2. Prove that if  $f$  is analytic in a neighborhood of  $z_0$  with a pole of order  $m$  at  $z_0$  and  $P$  is a polynomial of order  $n$ , then  $P \circ f$  has a pole of order  $mn$  at  $z_0$ .
3. Let  $\Omega \subset \mathbf{C}$  be a simply connected domain and let  $a \in \mathbf{C}$ . Suppose that  $f : \Omega \rightarrow \Omega$  is analytic,  $f(a) = a$ , and  $|f'(a)| = 1$ . Prove that  $f$  is a homeomorphism.
4. Let  $f$  be entire and nonconstant. Prove that there exists a point  $z_0 \in \mathbf{C}$  satisfying  $|f(z_0) - i| < \frac{1}{100}$ .
5. Identify all zeros and poles of the function  $f(z) = \frac{z}{\sin z}$  and determine their multiplicities.