Mathematical Physics **Qualifying Exam** **Name_____________**

Do any three of the 4 problems

1. For systems with the potential energy function $V(r)$ depending only on the distance r, the wave function can be expressed as the product of the radial wave function $R_{nl}(r)$ and the spherical harmonics $Y_{lm}(\theta,\varphi)$ where $Y_{lm}(\theta,\varphi)$ is the common eigenfunction of operators L^2 and L_z such that $L^2 Y_{lm}=l(l+1)\hbar^2 Y_{lm}$ and $L_z Y_{lm}=m\hbar\, Y_{lm}$. (a) Show that the radial wave equation is given by
$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dR_{nl}}{dr})+\{\frac{2m}{\hbar^2}[E-V(r)]-\frac{l(l+1)}{r^2}\}R_{nl}=0.$$
(b) By making the substitution $R_{nl}=u_{nl}(r)/r$, show that the redial wave function can be reduced to an effective one-dimensional Schrödinger equation
$d^2u_{nl}/dr^2+(2m/\hbar^2)[E-V_{eff}]u_{nl}=0$, where $V_{eff}=V(r)+l(l+1)\hbar^2/2mr^2$.
(c) For the case of the isotropic harmonic oscillator with $V(r)=m\omega^2r^2/2$, the effective one-dimensional radial wave equation can be written as
$$d^2u_{nl}/d\rho^2+(C-\rho^2-l(l+1)/\rho^2)u_{nl}=0$$
with the introduction of $\rho=\alpha r$, $\alpha=(m\omega/\hbar)^{1/2}$, and $C=2E/\hbar\omega$. By examining its asymptotic solutions at $r\to 0$ and $r\to\infty$ respectively, show that u_{nl} can be written as $u_{nl}(\rho)=\rho^{l+1}e^{-\rho^2/2}f(\rho)$ with $f(\rho)$ satisfying the differential equation
$$\rho(d^2f/d\rho^2)+2(l+1-\rho^2)(df/d\rho)-(2l+3-C)\rho f=0.$$
(d) Making a change of variable $\xi=\rho^2$, show that the differential equation for $f(\xi)$ is
$$\xi(d^2f/d\xi^2)+(l+3/2-\xi)(df/d\xi)-(1/4)(2l+3-C)f=0.$$
This is in the form of the well-known differential equation $xF''+(b-x)F'-aF=0$ whose solution is the confluent hypergeometric series
$$F(a,b,x)=\sum_s\frac{\Gamma(a+s)\Gamma(b)x^s}{\Gamma(a)\Gamma(b+s)\Gamma(s+1)}=1+\frac{ax}{b}+\frac{a(a+1)x^2}{b(b+1)2!}+\ldots.$$
It can be seen that F behaves as e^x for large x. Show that the requirement that u_{nl} must be normalizable leads to $E=(n+3/2)\hbar\omega$ where $n=2s+l$. Thus each energy state n is associated with several orbital angular momentum states l according to $l=n-2s=n,n-2,\ldots,(1$ or $0)$.

2. A general angular momentum operator $\vec{J}$ can be defined by the commutation relations of its components: $[J_x,J_y]=i\hbar J_z$. (a) Show that $[J^2,J_z]=0$. (b) Let the common eigenvector of J^2 and J_z be $|\lambda,m\rangle$ such that $J^2\,|\lambda,m\rangle=\lambda\hbar^2\,|\lambda,m\rangle$ and $J_z|\lambda,m\rangle=m\hbar\,|\lambda,m\rangle$. Show that $\lambda=j(j+1)$ and, for a given j, $m=-j,-j+1,\ldots,j$. (c) Show that possible values of j are $0,1/2,1,3/2,\ldots$. (d) Obtain the matrices J_x, J_y, and J_z in the representation of common eigenvectors of J^2 and J_z for $j=3/2$.

3. Consider a particle in a cylindrical box of radius a and length L. Show, using cylindrical coordinates, that the possible values of the energy are $E = (\hbar^2/2m)[(n\pi/L)^2 + (\varepsilon_{|m|\nu}/a)^2]$ while the corresponding eigenfunctions are $\varphi_{nm\nu}(r) = NJ_{|m|}(\varepsilon_{|m|\nu}r/a)e^{im\phi}\sin(n\pi z/L)$ with $m=0,\pm1,\pm2,\dots$, $\nu=1,2,3\dots$, and $\varepsilon_{|m|\nu}$ being the ν th root of the Bessel function of order $|m|$.
(Hints: $\nabla^2 = (1/r^2)\{(\partial/\partial r)[r^2(\partial/\partial r)] + (\partial^2/\partial\phi^2)\} + (\partial^2/\partial z^2)$ in the cylindrical coordinates; the Bessel function of order n satisfies the differential equation $d^2J_n/dr^2 + (1/r)(dJ_n/dr) + (1 - n^2/r^2)J_n = 0$.)

4. The Green's function $G(r)$ for a free particle is defined as the solution to the equation $(\hbar^2/2m)(\nabla^2 + k^2)G(r) = \delta(r)$. (a) Using $G(\vec{r}) = (2\pi)^{-3}\int G(\vec{q})e^{i\vec{q}\bullet\vec{r}}d^3\vec{q}$ and $\delta(\vec{r}) = (2\pi)^{-3}\int e^{i\vec{q}\bullet\vec{r}}d^3\vec{q}$, show that $G(\vec{q}) = (2m/\hbar^2)(k^2 - q^2)^{-1}$. (b) Determine $G(\vec{r})$.

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