Ph.D. Qualifying Examination in Probability

Department of Mathematics
University of Louisville
August 13, 2015, 9:00am–12:30pm

Do 3 problems from each section.

SECTION 1

Problem 1.

(a) State and prove Borel-Cantelli's Lemma.

(b) Suppose X is a random variable whose only values are *positive integers*.
Show that $EX = \sum_{n=1}^{\infty} P(X \geq n)$.

(c) Suppose $X, X_1, X_2, \ldots$ are i.i.d. random variables that take only *positive integer* values.
Suppose their common probability function is as follows:

$$\text{For each } n = 1, 2, \ldots, \;\; P(X = n) = C/n^3.$$

Here C is a normalizing constant – the positive constant such that $\sum_{n=1}^{\infty} C/n^3 = 1$.
Compute $P(X_n \geq n$ for infinitely many positive integers n).

Problem 2. Suppose that X_n are positive iid random variables with finite mean and $S_n = X_1 + X_2 + .. + X_n$. Show that $\max_{1 \leq k \leq n} X_k/S_n$ converges to 0 almost surely.

Problem 3. Thomas tosses a fair coin twice. Let us define a random variable X be the number of heads.

(a) Write the probability space of this experiment.

(b) Write the sigma algebra generated by all possible events.

(c) Let $\mathcal{G}$ be the sigma algebra generated by the events with only one head. Find $E(X|\mathcal{G})$.

Problem 4.

(a) State the Strong Law of Large Numbers.

(b) Let $f(t) = \dfrac{t}{t+1}$. Compute

$$\lim_{n\to\infty} \int_0^{\infty} \cdots \int_0^{\infty} f\left(\frac{x_1 + \ldots + x_n}{n}\right) 2^n \exp\{-2x_1 - \ldots - 2x_n\}\, dx_1 \cdots dx_n.$$

Make sure to show sufficient details and justify important steps.

SECTION 2

Problem 5.

(a) State Markov's inequality.

(b) Suppose that X is a random variable with moment generating function $M(t)$ which is defined for all real numbers t. Prove that $P(X \geq x) \leq e^{-tx}M(t)$ for $t \geq 0$.

(c) Suppose that Y has density function

$$f(y) = \frac{\theta^\alpha y^{\alpha-1}e^{-\theta y}}{\Gamma(\alpha)} \text{ for } y > 0$$

with $\theta > 0$ and $\alpha > 0$. Prove that $P\left(Y \geq \frac{3\alpha}{\theta}\right) \leq \left(\frac{3}{e^2}\right)^\alpha$.

Problem 6.

(a) Define a Standard Brownian Motion (SBM), $B(t),\ t \geq 0$ and show that $B(t)$ is a martingale.

(b) Show that the process $X_t = \frac{1}{\sqrt{c}}B(ct)$, $c > 0$ is also a SBM, whenever $B(t)$ is a SBM.

(c) Let B_n be a standard Brownian motion evaluated only at integer times $n = 1, 2, \ldots$. Show that the process $B_n^2 - n$ forms a martingale.

Problem 7. Suppose that the random variables X_n are defined on the same probability space and there is a constant c such that X_n converges in distribution to c.
Prove or disprove each of the following:

(a) X_n converges to c in probability;

(b) X_n converges to c a.s. .

Problem 8.

(a) State and prove the central limit theorem in $R^d,\ d \geq 1$.

(b) Suppose $X_1, X_2, \ldots$ are i.i.d random variables such that

$$P(X_1 = 2) = P(X_1 = -2) = 1/4 \text{ and } P(X_1 = 0) = 1/2.$$

Compute approximately

$$P\left(\sum_{k=1}^{10{,}000} X_k < 0 < \sum_{k=1}^{10{,}000} (X_k^2 - 2)\right).$$

All the details must be shown including computing integrals.

Ph.D. Qualifying Examination in Probability and Mathematical Statistics

Department of Mathematics
University of Louisville
August 14, 2013, 9:00am–12:30pm

Do all problems from both sections.

Section 1: Probability

1. Let $\Omega = (0, 1]$ and $A_n = \left\{\left(\frac{m}{2^n}, \frac{m+1}{2^n}\right] : m = 0, 1, \ldots, 2^n - 1\right\}$.
Let $\mathcal{F}_n$ be the σ-field generated by A_n.
(a) State a valid defintion of a σ-field.
(b) List all sets in $\mathcal{F}_1$ and list all sets in $\mathcal{F}_2$.
(c) Is $\displaystyle\bigcup_{n=1}^{\infty} \mathcal{F}_n$ a σ-field? If yes, prove it. If not, justify your answer.

2. Let $X, X_1, X_2, \ldots$ be random variables on a probability space. For each of the following statements, determine whether it is true or false. Then, give a proof of the statement if it is true or give a counterexample if it is false.
(a) If $\lim_{n\to\infty} \mathbb{E}\,|X_n - X| = 0$, then $X_n \to_p X$.
(b) If $X_n \to_p X$, then $X_n \to_d X$.
(c) If $X_n \to_d X$, then $X_n \to_p X$.

3. Suppose $X_1, X_2, \ldots$ are independent, $\mathbb{E}[X_n] = 0$ for all n, and $\left\{\mathbb{E}\left[X_n^4\right]\right\}_{n=1}^{\infty}$ is bounded.
(a) Show that $\displaystyle\mathbb{E}\left[\left(\sum_{i=1}^{n} X_i\right)^4\right] = \sum_{i=1}^{n} \mathbb{E}[X_i^4] + 6 \sum\sum_{1\le i<j\le n} \mathbb{E}[X_i^2]\ \mathbb{E}[X_j^2]$.
(b) Show that $\displaystyle\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| > \varepsilon\right) \le \frac{1}{\varepsilon^4 n^4}\mathbb{E}\left[\left(\sum_{i=1}^{n} X_i\right)^4\right]$
(c) Show that $\displaystyle\sum_{n=1}^{\infty} \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| > \varepsilon\right)$ converges.
(d) Prove that $\displaystyle\frac{1}{n}\sum_{i=1}^{n} X_n \to 0$ with probability 1.

4. Suppose that $\{X_k\}_{k=1}^{\infty}$ is an independent sequence of random variables such that X_k is uniformly distributed on the interval $\left(-1-2^{-k}, 1+2^{-k}\right)$.
(a) Compute $\mu_k = \mathbb{E}[X_k]$ and $\sigma_k^2 = \operatorname{Var}[X_k]$.
(b) Find the value $A \in (0, \infty)$ such that $\lim_{n\to\infty} \frac{\mathbb{E}\left[\sum_{k=1}^n X_k^4\right]}{n} = A$.
(c) Let $s_n^2 = \sum_{k=1}^{n} \sigma_k^2$. Find the value $B \in (0,\infty)$ such that $\lim_{n\to\infty} \frac{s_n^4}{n^2} = B$.
(d) Prove that $\frac{\sum_{k=1}^n (X_k - \mu_k)}{s_n} \to_d Z$ where Z is a standard normal random variable.

Section 2: Mathematical Statistics

5. Suppose that X has a Gamma(α,1) distribution with density

$$f_X(x|\alpha) = \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}, \quad x > 0.$$

(a) Find the probability density function $f_Y(y|\alpha)$ for the random variable $Y = \sqrt{X}$.
(b) Using part (a), show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
(c) A family of probability density functions is called an exponential family if it can be expressed as $f(x|\theta) = h(x)C(\theta)\exp\{W(\theta)t(x)\}$. Is $\{f_Y(y|\alpha)\}$ an exponential family? If yes, define θ and find $h(x)$, $C(\theta)$, $W(\theta)$, and $t(x)$? If not, justify your answer.

6. Let $X_1, X_2, \ldots, X_n$ be independent Normal random variables with unknown mean μ and variance 1.
(a) Show that $\sum_{i=1}^{n} X_i$ is a complete and sufficient statistic for μ.
(b) Find the uniformly minimum variance unbiased estimator of μ.
(c) Does the estimator in (b) attain the Cramér-Rao lower bound? Justify your answer.

7. Let $X_1, X_2, \ldots, X_n$ be a sample from probability mass function

$$P(X = k) = \begin{cases} 1/N & \text{if } k = 1, 2, \ldots, N, \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find the maximum likelihood estimator $\hat{N}$ of N.
(b) Show that $\mathbb{P}(\hat{N} > k) = 1 - \left(\frac{k}{N}\right)^n$ for $k = 1, \ldots, N$.
(c) For sample size $n = 2$, compute $\mathbb{E}[\hat{N}]$.

8. Let $X_1, X_2, \ldots, X_n$ be independent random variables following the Poisson distribution with probability mass function

$$P(X = k) = \begin{cases} \dfrac{\lambda^k e^{-\lambda}}{k!} & \text{if } k = 0, 1, 2, \ldots, \\ 0 & \text{otherwise} \end{cases} .$$

(a) Let $\bar{X}_n = \dfrac{\sum_{i=1}^n X_i}{n}$. Find the mean and standard deviation of $\bar{X}_n$.

(b) Let $Z_n = \dfrac{\bar{X}_n - \mathbb{E}[\bar{X}_n]}{\sqrt{\mathrm{Var}\left[\bar{X}_n\right]}}$. Using the fact that $Z_n \to_d Z$ where Z follows a standard normal distribution so that $P(Z_n < z_{1-\alpha}) \to 1 - \alpha$ as $n \to \infty$, construct an approximate (for large n) one-sided $100(1-\alpha)\%$ confidence interval for λ with lower confidence limit λ_ℓ based on the pivot Z_n.

(c) If $n = 10$, $\lambda = \frac{1}{10}$, and $\alpha = .05$, compute the exact coverage probability of the approximate 95% confidence interval proposed in (b). (Note that $z_{.95} \approx 1.645$.)

Ph.D. Qualifying Examination in Probability and Mathematical Statistics

Department of Mathematics
University of Louisville
Fall 2012

Do ALL the problems from both sections

1 Probability

1. (a) We toss a coin twice. Write the sample space Ω. Also, write the σ algebra generated by $\{HT\}$ and $\{TH\}$.
   (b) Show that, for two events A and B, if $A \subset B$, then $\mathbb{P}(P) \leq \mathbb{P}(B)$.
   (c) Prove that $\mathbb{P}(|X| > a) \leq \frac{E(X^2)}{a^2}$.

2. Consider a game with the following rule. One tosses a fair coin. If it is a tail, one gets nothing. If it is a head, one gets a certain amount of money, which follows the uniform distribution between \$0 and \$10. Find the distribution function of the money one gets from this game.

3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{X_n\}$ be a sequence of random variable defined on Ω. If X_n converges almost uniformly to X, then show that X_n converges almost everywhere to X.

4. Let X_n be independent and identically distributed random variables with $\mathbb{P}(X_n = 1) = \frac{1}{2}$ and $\mathbb{P}(X_n = -1) = \frac{1}{2}$. Show that $\sum_{j=1}^{n} X_j$ converges to 0 in probability.

2 Mathematical Statistics

1. The random variable X follows the exponential distribution $Exp(\beta)$ with pdf $f(x) = \frac{1}{\beta}\, e^{-\frac{1}{\beta}x} I_{\{0<x<\infty\}}$.

   (a) Find the pdf of $Y = X^{\frac{1}{\gamma}}$, $\gamma > 0$

   (b) A family of pdfs is called an exponential family if it can be expressed as $f(x/\theta) = h(x)\, C(\theta)\, exp\left(\sum_{i=1}^{k} W_i(\theta)\, t_i(x)\right)$. Does X make an exponential family? If yes, define θ and find $h(x)$, $C(\theta)$, $W_i(\theta)$, $t_i(x)$. If not, justify your answer.

   (c) Does Y make an exponential family? If yes, define θ and find $h(x)$, $C(\theta)$, $W_i(\theta)$, $t_i(x)$. If not, justify your answer.

2. Let $X_1, X_2, ..., X_n$ be a random sample from a distribution with pdf $f(x/\theta) = \frac{1}{\theta}\, e^{-\frac{1}{\theta}x}\, I_{\{0<x<\infty\}}$, where $0 < \theta < \infty$.

   (a) Find the maximum likelihood (ML) estimator of θ? Is this ML estimator of θ is a sufficient estimator of θ?

   (b) Using moment method find a consistent estimator of θ.

3. X_i $i = 1, 2, ..., n$ are i.i.d. from Bernoulli distribution with probability of success p.

   (a) Find a CSS for p.

   (b) Find the UMVUE of $p(1-p)$.

   (c) Notice that $X_1(1-X_2)$ is an unbiased estimator of $p(1-p)$. Using this fact, the CSS you found in part (a) and Lehman-Scheffe theorem, find the UMVUE.

4. Let $X_1, X_2, ..., X_n$ be a random sample of size n from a normal population with mean μ and variance σ^2. What is the likelihood ratio test of size α for testing the null hypothesis $H_o : \mu = \mu_o$ versus the alternative hypothesis $H_a : \mu \neq \mu_o$?

Ph.D. Qualifying Examination in Probability and Mathematical Statistics

Department of Mathematics
University of Louisville

Fall 2011

1 Probability

Do any three problems from this section.

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of events in $\mathcal{F}$ and

$$E = \{w \mid w \in E_n \text{ for infinitely many } \ n \in \mathbb{N}\}.$$

   (a) If $\displaystyle\sum_{n=1}^{\infty} \mathbb{P}(E_n) < \infty$, then prove that $\mathbb{P}(E) = 0$.

   (b) If $\displaystyle\sum_{n=1}^{\infty} \mathbb{P}(E_n) = \infty$ and the sequence $\{E_n\}$ consists of mutually independent events, then prove that $\mathbb{P}(E) = 1$.

2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X a random variable on Ω, and F the cdf of X. Then prove that F satisfies the followings:

   (a) If $x_1 \leq x_2$, then $F(x_1) \leq F(x_2)$ for all $x_1, x_2 \in \mathbb{R}$.

   (b) F is right continuous at each $x_o \in \mathbb{R}$.

   (c) $\displaystyle\lim_{x \to -\infty} F(x) = 0$ and $\displaystyle\lim_{x \to \infty} F(x) = 1$.

3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let X be a random variable and $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Then show that the composite function $f \circ X$ is a random variable.

4. State precisely a version of the Central Limit theorem. Give a proof of the Central Limit theorem.

2 Mathematical Statistics

Do any three problems from this section.

1. Suppose that $X_1, X_2, ..., X_n$ be a random sample from a population X having density function

$$f(x;\theta) = \begin{cases} \frac{\theta^2}{x^3}\, e^{-\frac{\theta}{x}} & \text{if } 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

where $\theta > 0$. It is known that $E(X_i) = \theta$ and $E\left(\frac{1}{X_i}\right) = \frac{2}{\theta}$ for each $i = 1, 2, ..., n$.

   (a) Determine the estimator of θ using moment method.

   (b) Compute the likelihood function for this random sample.

   (c) Find the maximum likelihood estimator of θ.

   (d) Show that $\displaystyle\sum_{i=1}^{n} \frac{1}{X_i}$ is a sufficient statistic for θ.

   (e) Find the Fisher information $I(\theta)$ in a single observation from this density function.

   (f) Using the maximum likelihood estimator based the sample, construct an approximate 90% confidence interval for θ.

   (g) Suppose $n = 8$ and $\displaystyle\sum_{i=1}^{8} \frac{1}{X_i} = 10$. Using these information on your confidence interval in (f), can you reject the null hypothesis $H_o : \theta = 1$ in favor of $H_a : \theta \neq 1$ at the significance level $\alpha = 0.10$?

   (h) Verify that the likelihood ratio critical region for testing the null hypothesis $H_o : \theta = \theta_o$ against $H_a : \theta \neq \theta_o$ has a critical region of the form

$$\left\{ (x_1, x_2, ..., x_n) \in \mathbb{R}^n_+ \;\middle|\; \left(\sum_{i=1}^{n} \frac{1}{X_i} \right)^{2n} \exp\left(-\theta_o \sum_{i=1}^{n} \frac{1}{X_i} \right) \leq k \right\}$$

   for some constant k.

2. Let $X_1, X_2, ..., X_n$ be a random sample from a normal distribution with mean zero and variance θ. Consider the class of estimators $c \sum_{i=1}^{n} X_i^2$.

   (a) What value of the constant c leads to an unbiased estimator?

   (b) Does this estimator achieve Cramer-Rao lower bound?

   (c) Find the unique value of c which minimizes the mean squared error

   $$E_\theta \left(\left(c \sum_{i=1}^{n} X_i^2 - \theta \right)^2 \right)$$

   uniformly.

3. Answer both parts of this question.

   (a) Let $X_1, X_2, ..., X_n$ be a random sample from a continuous population X with a distribution function $F(x;\theta)$. If $F(x;\theta)$ is monotone in θ, then show that the statistic

   $$Q = -2 \sum_{i=1}^{n} \ln \left(1 - F\left(X_i;\theta\right) \right)$$

   has a chi-square distribution with $2n$ degrees of freedom.

   (b) If $X_1, X_2, ..., X_n$ is a random sample from a population with density

   $$f(x;\theta) = \begin{cases} \theta\, x^{\theta-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

   where $\theta > 0$ is an unknown parameter, what is a $100(1-\alpha)\%$ confidence interval for θ?

4. Let $X_1, X_2, ..., X_n$ be a random sample of size n from a normal population with mean μ and **known variance** σ^2. What is the likelihood ratio test of size α for testing the null hypothesis $H_o : \mu = \mu_o$ versus the alternative hypothesis $H_a : \mu \neq \mu_o$?

Probablity and Mathematical Statistics Qualifier Exam

May 18, 2007

Remark : Choose three problems in 1-4, and choose three problems in 5-8

Name :

Problem 1: Let $(\Omega, \mathcal{F}, \mathbf{P}) = ([0,1), Borel, Lebesgue)$. For $n = 0, 1, 2, 3...$ let $Q_n = \sigma([\frac{k}{2^n}, \frac{k+1}{2^n}) : 0 \leq k < n)$ be the dyadic partitions of $[0,1)$. Let $X(s) = s, , 0 \leq s < 1$.
(a)Find the equation of and sketch $E(X|Q_0)$,$E(X|Q_1)$ and $E(X|Q_2)$.

(b)Let $Y_n = E(X|Q_n)$. Show that $E(Y_6|Q_4) = Y_4$. Find $E(Y_4|Q_6)$.

Problem 2:
(a) Suppose that $\Omega \in \mathcal{F}$ and that $A, B \in \mathcal{F}$ implies $A - B \in \mathcal{F}$. Show that $\mathcal{F}$ is a field.

(b) Suppose that $\Omega \in \mathcal{F}$ and that $\mathcal{F}$ is closed under the formation of complements and finite disjoint unions. Find a counterexample which shows that $\mathcal{F}$ need not be a field.

Problem 3:
(a) Let $X, Y \in L^1$. If X and Y are independent, show that $XY \in L^1$. Give an example to show XY need not be in L^1 in general.

(b) Let X_n be i.i.d. random variables with $P(X_n = 1) = \frac{1}{2}$ and $P(X_n = -1) = \frac{1}{2}$. Show that $\frac{1}{n}\sum_{j=1}^{n} X_j$ converges to 0 in probability.

Problem 4:
(a) Suppose that $X_1, X_2, \ldots$ is an independent sequence and Y is measurable $\sigma(X_n, X_{n+1}, \ldots)$ for each n. Show that there exists a constant a such that $P(Y = a) = 1$.

(b) Prove for integrable X that

$$E[X] = \int_0^\infty P(X > t)\ dt - \int_{-\infty}^0 P(X < t)\ dt.$$

Problem 5:
(a) f $Z \sim N(0, 1)$, prove the following.

$$P(|Z| \geq t) \geq \sqrt{\frac{2}{\pi}} \frac{t}{t^2 + 1} e^{-\frac{t^2}{2}}.$$

(b) Let $X_1, X_2, ..., X_n$ be an i.i.d. sample from $\mathrm{B}(n, p)$. Show that there is no unbiased estimator of $g(p) = 1/p$.

(c) Suppose there are n AA Financial bonds, and let X_i denote the spread return, in a given month, of the ith bond, $i = 1, \ldots, n$. Suppose that they are all independent and follows normal distribution with a common mean μ but different variances σ_i^2. Show that the joint distribution of $X = (X_1, X_2, \ldots, X_n)$ makes an exponential family and find $C(x), h(x), w_i(x)$ and $T_i(x)$.

Problem 6: Let $X_1, X_2, ..., X_n$ be i.i.d. random sample with density

$$f(x|\mu, \sigma) = \frac{1}{\sigma} \exp(-\frac{x - \mu}{\sigma}) I_{(\mu, \infty)}(x),$$

where $\sigma > 0$.
(a) Find a two-dimensional sufficient statistic for (μ, σ).

(b) Find the distribution of $\frac{n}{\sigma}(X_{(1)} - \mu)$.

(c) Find the distribution of $\sum_{i=1}^n (X_i - X_{(1)})/\sigma$.

(d) Using (2) and (3), check that the sufficient statistic you found in (a) is also complete.

Problem 7:
(a) Let $X_1, X_2, ..., X_n$ be i.i.d. from Uniform$(0, \theta)$. Find the Complete Sufficient Statisitcs of θ. Also, find the UMVUE of θ.

(b) Let $X_1, X_2, ..., X_n$ be i.i.d. from Poisson(λ). Find a Complete Sufficient Statistic of λ. Also, find the UMVUE of $\eta = P(X_1 = a)$.

Problem 8:
(a) Let $X_1, X_2, ..., X_n$ be i.i.d. from a distribution having p.d.f. of the form $f(x) = \theta x^{\theta-1} I_{(0,1)}(x)$. Find the Rejection Region of the most poweful test for $H_0 : \theta = 1$ versus $H_1 : \theta = 2$

(b) Let $X_1, X_2, ..., X_n$ be i.i.d. from a distribution having p.d.f. of the form $f(x) = e^{-(x-\theta)} I_{[\theta,\infty)}(x)$. Find the likelihood ratio test for $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$.

NAME:

INSTRUCTIONS: Clearly mark the answers and show all the work. You may use calculators when appropriate. Solve three problems from each part. Clearly marke the ones you are NOT solving

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PART ONE: PROBABILLITY
SOLVE THREE OUT OF THE NEXT FOUR PROBLEMS

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1. (50 points) This question consists of three parts.

   (a) (15 points) Let X_i be a sequence of random variables such that

   $$\lim_{n\to\infty} \frac{VarS_n}{n^2} = 0, \tag{1}$$

   where $S_n = \sum_{i=1}^n X_i$. Show that

   $$\frac{S_n - ES_n}{n} \xrightarrow{P} 0 \quad \text{as } n \to \infty. \tag{2}$$

   (b) (15 points) Now, suppose that we replace the condition (1) with the assumption that the X_i are pairwise uncorrelated and satisfy $\sup_i EX_i^2 < \infty$. Show that the result (2) above also holds under these alternative assumptions.

   (c) (20 points) Assuming that the X_i are independent random variables such that $\sup_i EX_i^4 < \infty$, show that the convergence in probability in (2) can be strengthened to convergence almost surely.

2. (50 points points) Given a probability space $(\Omega, \mathcal{F}_0, P)$, an $\mathcal{F}_0$-measureable random variable X and another σ-field $\mathcal{F} \subset \mathcal{F}_0$, the ***conditional expectation of*** X ***given*** $\mathcal{F}$ is defined to be any random variable Y which is $\mathcal{F}$-measureable and satisfies

$$\int_A X dP = \int_A Y dP$$

for all $A \in \mathcal{F}$.

(a) (15 points) To warm up, consider how this definition relates to the one taught in undergraduate probability. Specifically, suppose that $\Omega_1, \Omega_2, \ldots$ is a finite or infinite partition of Ω into disjoint sets each of which has positive probability (with respect to P), and let $\mathcal{F} = \sigma(\Omega_1, \Omega_2, \ldots)$ the σ-field generated by these sets. Then show that on each Ω_i,

$$E(X \mid \mathcal{F}) = \frac{E(X; \Omega_i)}{P(\Omega_i)}.$$

(b) (15 points) Let $\mathcal{F}_1 \subset \mathcal{F}_2$ be two σ-fields on Ω. Then show that

1. $E(E(X \mid \mathcal{F}_1) \mid \mathcal{F}_2) = E(X \mid \mathcal{F}_1)$, and
2. $E(E(X \mid \mathcal{F}_2) \mid \mathcal{F}_1) = E(X \mid \mathcal{F}_1)$.

(c) (20 points) Let $\Omega = \{a, b, c\}$. Give an example of $P, \mathcal{F}_1, \mathcal{F}_2$, and X in which

$$E(E(X \mid \mathcal{F}_1) \mid \mathcal{F}_2) \neq E(E(X \mid \mathcal{F}_2) \mid \mathcal{F}_1).$$

3. (50 points) This problem consists of two parts.

   (a) (25 points) Let X be $N(0,1)$ random variable. Let

$$M(s) = E\{e^{s\,X}\} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(s\,x - \frac{x^2}{2}\right) dx.$$

   Show that $M(s) = e^{s^2/2}$.

   (b) (25 points) Show that for any positive integer n we have $E\{X^{2n+1}\} = 0$ and

$$E\{X^{2n}\} = \frac{(2n)!}{2^n\,n!} = (2n-1)(2n-3)\ldots 3 \cdot 1.$$

   (Hint: Note that $e^{s^2/2} = \sum_{k=0}^{\infty} \frac{s^{2k}}{2^k\,k!}$

4. (50 points) This question consists of two parts.

   (a) (25 points) Show that for any c.d.f. F and any $a \geq 0$

$$\int [F(x+a) - F(x)]dx = a$$

   (b) (25 points) Let X be a random variable with range $\{0,1,2,\ldots\}$. Show that if $EX < \infty$ then

$$EX = \sum_{i=1}^{\infty} P(X \geq i)$$

PART TWO: MATHEMATICAL STATISTICS

SOLVE THREE OUT OF THE NEXT FOUR PROBLEMS

5. (50 points) Let $U_1, \ldots, U_n$ be i.i.d. random variables having uniform distribution on [0,1] and $Y_n = (\prod_{i=1}^n U_i)^{-1/n}$. Show that

$$\sqrt{n}(Y_n - e) \xrightarrow{D} N(0, e^2)$$

where $e = \exp(1)$.

6. (50 points) Let ϕ be a UMP test of level $\alpha \in (0,1)$ for testing simple hypothesis P_0 vs P_1. If β is the power of the test, show that $\beta \geq \alpha$ with equality if and only if $P_0 = P_1$.

7. (50 points) Let $X_1, \ldots, X_n$ be independently and identically distributed with density

$$f(x, \theta) = \frac{1}{\sigma} \exp\left\{-\frac{x-\mu}{\sigma}\right\}, \; x \geq \mu,$$

where $\theta = (\mu, \sigma^2), -\infty < \mu < \infty, \sigma^2 > 0$.

(a) Find maximum likelihood estimates of μ and σ^2.

(b) Find the maximum likelihood estimate of $P_\theta(X_1 \geq t)$ for $t > \mu$.

8. (50 points) Let $X_1, \ldots, X_n$ be iid from Bernoulli distribution with unknown probability of success $P(X_1 = 1) = p \in (0, 1)$.

   (a) (20 points) Show that $S = \sum_{i=1}^{n} X_i$ is a complete and sufficient statistic.

   $$\text{(Hint: } \sum_{k=0}^{n} \binom{n}{k} g(k) p^k (1-p)^{n-k} = (1-p)^n \left(\sum_{k=0}^{n} \binom{n}{k} g(k) \zeta^k \right) \text{ where } \zeta = \frac{p}{1-p}.\text{)}$$

   (b) (30 points) Find UMVUE for p^m when $m \leq n$ is a positive integer.

Done !

Preliminary Examination **Probability and Statistics** **May 23, 2003**

Instruction: You are required to complete this examination within 3 hours. `Do only 5 problems, at least 2 from each group.` Work on one side of the page only. Start each problem on a new page. In order to receive full credit, answer each problem completely and must show all work.

GROUP A

1. If $A_1, A_2, ..., A_n, ...$ is a sequence of events in a sample space S such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$, then prove that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

Why this result is called the continuity theorem for the probability measure?

2. Consider the function

$$f(x,y) = \begin{cases} 2 & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that f is a bivariate density function.

Let the random variables X and Y have f as their joint density function. Determine

(b) the marginal density, $f_1(x)$, of X;

(c) the conditional density, $f(y/x)$, of Y given that $X = x$;

(d) the mean of this conditional density, that is $E(Y/X = x)$;

(e) the variance of this conditional density, that is $Var(Y/X = x)$.

3. Answer any four from the followings:

(a) State precisely Chebychev's inequality.

(b) State precisely the weak law of large numbers.

(c) State precisely Kolmogorov's inequality.

(d) State precisely the strong law of large numbers.

(e) State precisely the central limit theorem.

4. Answer the followings:

(a) Suppose X is a nonnegative random variable with mean $E(X)$. Then show that

$$P(X \geq t) \leq \frac{E(X)}{t}$$

for all $t > 0$.

(b) Suppose $X_1, X_2, ...$ is a sequence of random variables defined on a sample space S. Define the notion of convergence in probability of the sequence random variables $\{X_n\}$ to X.

(c) Suppose the random variable X and the sequence $X_1, X_2, ...,$ of random variables are defined on a sample space S. Define the notion of convergence almost surely of the sequence random variables $\{X_n\}$ to X.

(d) Suppose X is a random variable with cumulative density function $F(x)$ and the sequence $X_1, X_2, ...$ of random variables with cumulative density functions $F_1(x), F_2(x), ...,$ respectively. Define the notion of convergence in distribution of the sequence random variables $\{X_n\}$ to X.

GROUP B

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5. Answer the followings:

(a) What is the basic principle of the maximum likelihood estimation?

(b) Define an estimator for a population parameter θ.

(c) Define an unbiased estimator of a parameter θ.

(d) Define a sufficient estimator of a parameter θ.

(e) Let $X_1, X_2, ..., X_n$ be a random sample from a normal population with known mean μ and unknown variance $\sigma^2 > 0$. Determine the maximum likelihood estimators of μ and σ^2?

6. Observations $y_1, y_2, ..., y_n$ are assumed to come from a model with $E(Y_i/x_i) = \alpha + \beta x_i$, where α and β are unknown parameters, and $x_1, x_2, ..., x_n$ are given constants. Derive the least squares estimates of the parameters α and β in terms of the sum of squares S_{xx} and S_{yy}, and the sum of cross product S_{xy}, where $S_{xy} = \sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})$.

7. Answer the followings:

(a) Define a $100(1-\alpha)\%$ confidence interval for a parameter θ.

(b) Define an interval estimator of a parameter θ.

(c) Define a pivotal quantity for a parameter θ.

(d) If $X_1, X_2, ..., X_n$ is a random sample from a population with density

$$f(x;\theta) = \begin{cases} \theta\, x^{\theta-1} & \text{if } 0 < x < 1 \\ \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is an unknown parameter, then derive a $100(1-\alpha)\%$ approximate confidence interval for θ assuming the sample size is large.

8. Answer the followings:

(a) Define a hypothesis test.

(b) Define the power function of a hypothesis test

(c) State precisely the Neyman-Pearson Lemma.

(d) Let $X_1, X_2, ..., X_n$ be a random sample from a normal population with mean μ and known variance σ^2. Derive the likelihood ratio test of size α for testing the null hypothesis $H_o : \mu = \mu_o$ versus the alternative hypothesis $H_a : \mu \neq \mu_o$?