Ph.D. Qualifying Examination in Probability

Department of Mathematics
University of Louisville
January 4, 2018, 9:00am–12:30pm

Do 3 problems from each section.

SECTION 1

Problem 1.

(a) State and prove the Lindeberg Central Limit Theorems for triangular arrays.

(b) Prove that the Lyapounov's condition for some $\delta > 0$ implies the Lindeberg's condition.

(c) Suppose $X_1, X_2, \ldots$ are independent random variables such that for each $k \geq 1$, $|X_k| \leq C$ almost surely and $EX_k = 0$. Suppose that $\sum_{k=1}^{\infty} EX_k^2 = \infty$. You may also assume that $EX_k^2 > 0$, for every $k \geq 1$. Prove that for some sequence of positive numbers $s_n, n = 1, 2 \ldots$, one has that

$$S_n/s_n \Rightarrow N(0,1) \text{ as } n \to \infty, \text{ where } S_n = X_1 + X_2 + \ldots + X_n.$$

Problem 2. Let $X, X_1, X_2, \ldots$ be random variables on some probability space $(\Omega, \mathcal{F}, P)$.

(a) Show that if $X_n \Rightarrow X$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous then $g(X_n) \Rightarrow g(X)$.

(b) Show that the following three statements are equivalent:

1).
$$X_n \Rightarrow X,$$

2).
$$E(f(X_n)) \to E(f(X)), \text{ for all bounded continuous functions } f,$$

3).
$$P(X_n \in A) \to P(X \in A), \text{ for any Borel set } A \subseteq \mathbb{R}^k \text{ such that } P(X \in \partial A) = 0,$$

where ∂A is the boundary of A.

Problem 3. Let $X, X_1, X_2, \ldots$ be random variables on some probability space $(\Omega, \mathcal{F}, P)$.

(a) Show that $X_n \to X$ a.s. whenever $\sum_{n=1}^{\infty} E(|X_n - X|^r) < \infty$ for some $r > 0$.

(b) Suppose a is a real value such that $EX_n \to a$ and $Var(X_n) \to 0$ as $n \to \infty$. Show that $X_n \to a$ in probability.

(c) Suppose that $X_1, X_2, \ldots$ is a sequence of uncorrelated variables with zero means and uniformly bounded variances. Show that

$$\frac{\sum_{i=1}^{n} X_i}{n} \to 0 \text{ in mean square, i.e. } L_2 - \text{convergence}.$$

Problem 4. Suppose Ω is a nonempty set and $\mathcal{F}$ is a family of all subsets which are either finite or cofinite (i.e. have finite complements) in an infinite Ω.

(a) Show that $\mathcal{F}$ is a field, but not a σ-field.

(b) Define P on $\mathcal{F}$ by taking $P(A)$ to be 0 or 1 as A is finite or cofinite. Show that P is finitely additive.

(c) Show that this P is not countably additive if Ω is countably infinite.

SECTION 2

Problem 5. Suppose $X_1, X_2, \ldots$ are i.i.d. random variables with $E|X_1| < \infty$. Suppose N is a bounded stopping time with respect to a filtration $\mathcal{F}_n$. Define the partial sum $S_n = X_1 + X_2 + \ldots + X_n$.

(a) Show that $(S_n - n\mu)$, $n = 1, 2, \ldots$ is a martingale relative to the filtration $\mathcal{F}_n$, where $\mu = EX_1$.

(b) Show that $ES_N = EN \cdot EX_1$.

(c) Suppose $X_1, X_2, \ldots$ and N are as stated above, but with the additional properties that $EX_1 = 0$ and $EX_1^2 = \sigma^2 < \infty$. Show that $(S_n^2 - n\sigma^2)$, $n = 1, 2, \ldots$ is a martingale relative to the filtration $\mathcal{F}_n$ and $Var(S_N) = \sigma^2 \cdot EN$.

Problem 6. Let $X, X_1, X_2, \ldots$ be independent random variables with zero means and finite variances on some probability space $(\Omega, \mathcal{F}, P)$, and let $S_n = X_1 + X_2 + \ldots + X_n$.

(a) Show that $P(\max_{1\le i\le n} |S_i| \ge \epsilon) \le \frac{1}{\epsilon^2} \sum_{i=1}^n var(X_i)$ for $\epsilon > 0$. You can freely use the Doob-Kolmogorov's inequality: If Y_n is a martingale such that $EY_n^2 < \infty$ for all n, then

$$P(\max_{1\le i\le n} |Y_i| \ge \epsilon) \le \frac{1}{\epsilon^2} E(Y_n^2) \text{ for } \epsilon > 0.$$

(b) Assume now that $X_1, X_2, \ldots$ are random variables with zero means and finite variances and that X_n is a martingale. Show that

1).

$$E(X_{n+r} - X_n)^2 = \sum_{k=1}^{r} E[(X_{n+k} - X_{n+k-1})^2]$$

and

2).

$$\text{if } \sum_{n=1}^{\infty} E[(X_n - X_{n-1})^2] < \infty, \ X_n \text{ converges with probability 1.}$$

Problem 7.

(a) Define a Standard Brownian Motion (SBM), $(B(t),\ t \geq 0)$ and show that $B(t)$ is a Gaussian process. Compute its mean and covariance.

(b) Which of the following processes are SBM, given that $(B(t), t > 0)$ is a SBM? Justify your answers completely.

1).

$$X(t) = \begin{cases} 0, & \text{if } t = 0, \\ tB(\frac{1}{t}), & t > 0. \end{cases}$$

2). $X(t) = \sqrt{t}B(1)$.

(c) Show that almost surely, $\lim_{t\to\infty} \frac{B(t)}{t} = 0$, where $B(t)$ is a SBM.

Problem 8.

(a) Suppose that $X, X_1, X_2, \ldots$ is a sequence of random variables, each having normal distribution and such that $X_n \Rightarrow X$. Hence X has a normal distribution, possibly degenerate. For each $n \geq 1$, let (X_n, Y_n) be a pair of random variables having a bivariate normal distribution. Suppose that $X_n \to X$ in probability and $Y_n \to Y$ in probability. Show that (X, Y) has a bivariate normal distribution.

(b) Suppose now that you have two random variables X and Y, each one with normal distribution and in addition, they are uncorrelated. Prove or disprove that X and Y are independent random variables.

(c) Suppose now that X and Y have a bivariate normal distribution and they are uncorrelated. Prove or disprove that X and Y are independent normal random variables.

You can freely use that the joint probability density functions is

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right\},$$

where $\mu_X, \mu_Y \in \mathbb{R}, \sigma_X^2, \sigma_X^2 > 0, \rho \in (-1,1)$ are all constants. Here ρ is the coefficient of correlation.

Ph.D. Qualifying Examination in Probability

Department of Mathematics
University of Louisville
August 12, 2016, 9:00am–12:30pm

Do 3 problems from each section.

SECTION 1

Problem 1.

(a) State and prove Borel-Cantelli's Lemma.

(b) Suppose there is an infinite sequence of warped coins. Suppose that for coin n in the sequence, the probability of "heads" is $1/\sqrt{n}$. Suppose that these coins are each tossed once, in sequence (i.e. coin 1 is tossed, then coin 2 is tossed, then coin 3, and so on), and of course these tosses have no influence on each other. Compute the probability that infinitely many of the coins come up heads.

(c) Same context as in part (b).

  (1) For each $n \geq 1$, let B_n denote the event that coins n and $n+1$ each come up heads. Show that the events B_1 and B_2 are NOT independent.

  (2) Show that with probability 1, there will be infinitely many cases of two consecutive heads.

Problem 2.

(a) Suppose $X_1, X_2, \ldots ...$ is a sequence of **independent** random variables such that for each $k \geq 1$, $EX_k = 0$ and $EX_k^2 \leq k^{0.9}$. The random variables here are *not* assumed to be identically distributed. Prove that $\overline{X} = S_n/n$ converges in probability to 0 as $n \to \infty$.

(b) Show that if $X_n \to a$ in probability, where a is a real number, then $X_n \Rightarrow a$, that is $X_n \to a$ in distribution. Of course, $X_1, X_2, \ldots$ are not necessarily iid. You can freely use the fact that if X, Y are random variables, a is a real number and $\epsilon > 0$, then

$$P(Y \leq a) \leq P(X \leq a + \epsilon) + P(|Y - X| > \epsilon).$$

(c) Is the converse in part (b) true? Prove or disprove.

Problem 3. Thomas tosses a fair coin twice. Let us define a random variable X be the number of heads.

(a) Write the probability space of this experiment.

(b) Write the sigma algebra generated by all possible events.

(c) Let $\mathcal{G}$ be the sigma algebra generated by the events with only one head. Find $E(X|\mathcal{G})$.

Problem 4.

(a) State the Strong Law of Large Numbers.

(b) You are given the following statement: If $X_n \Rightarrow X$ and $Y_n \Rightarrow c$, where c is a constant, then $(X_n, Y_n) \Rightarrow (X, c)$. Prove that $X_n Y_n \Rightarrow Xc$.

(c) Let $X_1, X_2, \ldots$ be i.i.d. random variables with $EX_1 = 0$ and $0 < EX_1^2 < \infty$. Prove that

$$\frac{\sum_{k=1}^n X_k}{\sqrt{\sum_{k=1}^n X_k^2}} \Rightarrow N(0,1) \ \ as \ \ n \to \infty.$$

SECTION 2

Problem 5.

(a) State Markov's inequality.

(b) Suppose that X is a random variable with moment generating function $M(t)$ which is defined for all real numbers t. Prove that $P(X \geq x) \leq e^{-tx} M(t)$ for $t \geq 0$.

(c) Suppose that Y has density function

$$f(y) = \frac{\theta^\alpha y^{\alpha-1} e^{-\theta y}}{\Gamma(\alpha)} \text{ for } y > 0$$

with $\theta > 0$ and $\alpha > 0$. Prove that $P\left(Y \geq \frac{3\alpha}{\theta}\right) \leq \left(\frac{3}{e^2}\right)^\alpha$.

Problem 6.

(a) Define a Standard Brownian Motion (SBM), $B(t),\ t \geq 0$ and show that $B(t)$ is a martingale.

(b) Show that the process $X_t = \frac{1}{\sqrt{c}} B(ct)$, $c > 0$ is also a SBM, whenever $B(t)$ is a SBM.

(c) Let B_n be a standard Brownian motion evaluated only at integer times $n = 1, 2, \ldots$. Show that the process $B_n^2 - n$ forms a martingale.

Problem 7. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Suppose $\mathcal{F}_1 \subseteq \mathcal{F}_2$ are σ-fields ($\subseteq \mathcal{F}$).

(a) Prove $E(aX + bY|\mathcal{F}) = aE(X|\mathcal{F}) + bE(Y|\mathcal{F})$, where a and b are constants.

(b) Prove that $E(E(X|\mathcal{F}_1)|\mathcal{F}_2) = E(X|\mathcal{F}_1)$.

(c) Prove that $E(E(X|\mathcal{F}_2)|\mathcal{F}_1) = E(X|\mathcal{F}_1)$.

Problem 8.

(a) State and prove the central limit theorem in R^d, $d \geq 1$.

(b) Suppose $(X_1, Y_1), (X_2, Y_2), \ldots$ are i.i.d. "row" random vectors such that:

$$P(X_1 = 2, Y_1 = 0) = 1/3,$$

$$P(X_1 = -1, Y_1 = \sqrt{3}) = 1/3,$$
$$P(X_1 = -1, Y_1 = -\sqrt{3}) = 1/3.$$

Evaluate

$$\lim_{n\to\infty} P\left((X_1 + X_2 + \ldots + X_n)^2 + (Y_1 + Y_2 + \ldots + Y_n)^2 \leq n\right).$$

Make sure you explicitly write the density function. All the work should be presented for full credit.