Ph.D. Qualifying Examination in Probability

Department of Mathematics
University of Louisville
August 12, 2016, 9:00am–12:30pm

Do 3 problems from each section.

SECTION 1

Problem 1.

(a) State and prove Borel-Cantelli's Lemma.

(b) Suppose there is an infinite sequence of warped coins. Suppose that for coin n in the sequence, the probability of "heads" is $1/\sqrt{n}$. Suppose that these coins are each tossed once, in sequence (i.e. coin 1 is tossed, then coin 2 is tossed, then coin 3, and so on), and of course these tosses have no influence on each other. Compute the probability that infinitely many of the coins come up heads.

(c) Same context as in part (b).

  (1) For each $n \geq 1$, let B_n denote the event that coins n and $n+1$ each come up heads. Show that the events B_1 and B_2 are NOT independent.

  (2) Show that with probability 1, there will be infinitely many cases of two consecutive heads.

Problem 2.

(a) Suppose $X_1, X_2, \ldots ...$ is a sequence of **independent** random variables such that for each $k \geq 1$, $EX_k = 0$ and $EX_k^2 \leq k^{0.9}$. The random variables here are *not* assumed to be identically distributed. Prove that $\overline{X} = S_n/n$ converges in probability to 0 as $n \to \infty$.

(b) Show that if $X_n \to a$ in probability, where a is a real number, then $X_n \Rightarrow a$, that is $X_n \to a$ in distribution. Of course, $X_1, X_2, \ldots$ are not necessarily iid. You can freely use the fact that if X, Y are random variables, a is a real number and $\epsilon > 0$, then

$$P(Y \leq a) \leq P(X \leq a + \epsilon) + P(|Y - X| > \epsilon).$$

(c) Is the converse in part (b) true? Prove or disprove.

Problem 3. Thomas tosses a fair coin twice. Let us define a random variable X be the number of heads.

(a) Write the probability space of this experiment.

(b) Write the sigma algebra generated by all possible events.

(c) Let $\mathcal{G}$ be the sigma algebra generated by the events with only one head. Find $E(X|\mathcal{G})$.

Problem 4.

(a) State the Strong Law of Large Numbers.

(b) You are given the following statement: If $X_n \Rightarrow X$ and $Y_n \Rightarrow c$, where c is a constant, then $(X_n, Y_n) \Rightarrow (X, c)$. Prove that $X_n Y_n \Rightarrow Xc$.

(c) Let $X_1, X_2, \ldots$ be i.i.d. random variables with $EX_1 = 0$ and $0 < EX_1^2 < \infty$. Prove that

$$\frac{\sum_{k=1}^n X_k}{\sqrt{\sum_{k=1}^n X_k^2}} \Rightarrow N(0,1) \ \ as \ \ n \to \infty.$$

SECTION 2

Problem 5.

(a) State Markov's inequality.

(b) Suppose that X is a random variable with moment generating function $M(t)$ which is defined for all real numbers t. Prove that $P(X \geq x) \leq e^{-tx} M(t)$ for $t \geq 0$.

(c) Suppose that Y has density function

$$f(y) = \frac{\theta^\alpha y^{\alpha-1} e^{-\theta y}}{\Gamma(\alpha)} \text{ for } y > 0$$

with $\theta > 0$ and $\alpha > 0$. Prove that $P\left(Y \geq \frac{3\alpha}{\theta}\right) \leq \left(\frac{3}{e^2}\right)^\alpha$.

Problem 6.

(a) Define a Standard Brownian Motion (SBM), $B(t),\ t \geq 0$ and show that $B(t)$ is a martingale.

(b) Show that the process $X_t = \frac{1}{\sqrt{c}} B(ct)$, $c > 0$ is also a SBM, whenever $B(t)$ is a SBM.

(c) Let B_n be a standard Brownian motion evaluated only at integer times $n = 1, 2, \ldots$. Show that the process $B_n^2 - n$ forms a martingale.

Problem 7. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Suppose $\mathcal{F}_1 \subseteq \mathcal{F}_2$ are σ-fields ($\subseteq \mathcal{F}$).

(a) Prove $E(aX + bY|\mathcal{F}) = aE(X|\mathcal{F}) + bE(Y|\mathcal{F})$, where a and b are constants.

(b) Prove that $E(E(X|\mathcal{F}_1)|\mathcal{F}_2) = E(X|\mathcal{F}_1)$.

(c) Prove that $E(E(X|\mathcal{F}_2)|\mathcal{F}_1) = E(X|\mathcal{F}_1)$.

Problem 8.

(a) State and prove the central limit theorem in $R^d,\ d \geq 1$.

(b) Suppose $(X_1, Y_1), (X_2, Y_2), \ldots$ are i.i.d. "row" random vectors such that:

$$P(X_1 = 2, Y_1 = 0) = 1/3,$$

$$P(X_1 = -1, Y_1 = \sqrt{3}) = 1/3,$$
$$P(X_1 = -1, Y_1 = -\sqrt{3}) = 1/3.$$

Evaluate

$$\lim_{n\to\infty} P\left((X_1 + X_2 + \ldots + X_n)^2 + (Y_1 + Y_2 + \ldots + Y_n)^2 \leq n\right).$$

Make sure you explicitly write the density function. All the work should be presented for full credit.