forcing occurs in C^n until the cut-vertex v is turned black. Now, if v is the last vertex on C^{n-1} to be turned black, then it's clear that at most two vertices in $< V(C^{n-1} \cup C^n) >$ are turned black at a time - since, obviously, with any $Z(B_n)$-set at most two vertices are turned black on any cycle C^i at each step. On the other hand, if v is turned black before C^{n-1} is turned entirely black, then v, being a neighbor to at least two white vertices, can not force until C^{n-1} is turned entirely black. We've thus shown that $Z_m \cap V(C^{n-1} \cup C^n)$ contains at most two forcing vertices for any m or, equivalently, at most two vertices in $< V(C^{n-1} \cup C^n) >$ are turned black at a time. $\square$

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REMARKS ON THE DOMINATION NUMBER OF GRAPHS

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ABSTRACT. This paper consists of two loosely related notes on the domination number of graphs. In the first part, we provide a new upper bound for the domination number of d-regular graphs. Our bound is the best known for $d \geq 6$. In the second part, we compute the domatic number of certain Kneser graphs.

1. INTRODUCTION

Let $G = (V, E)$ be a graph and $X \subseteq V$. X is a dominating set if for every $v \in V$ the closed neighborhood of v and X are not disjoint. X is a total dominating set if for every $v \in V$ the open neighborhood of v and X are not disjoint.

The domination number $\gamma(G)$ is the minimum size of a dominating set and the total domination number $\gamma_t(G)$ is the minimum size of a total dominating set. It is clear from the definition that $\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G)$.

A domatic partition of a graph G is a partition of V into sets $V_1, \ldots, V_k$ such that each V_i is a dominating set of G. The domatic number $\mathrm{dom}(G)$ of a graph G is the maximum size of a domatic partition. From the definition it is immediate that $\mathrm{dom}(G) \leq \frac{|V(G)|}{\gamma(G)}$.

Extensive research has been done to find upper bounds for the domination number of graphs of given minimum degree. Alon and Spencer [2] provide a probabilistic proof for the following theorem.

Theorem 1.1. *Let G be a graph on n vertices with minimum degree $\delta > 1$. Then*

$$\gamma(G) \leq \frac{1 + \ln(\delta + 1)}{\delta + 1} \cdot n \tag{1}$$

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Arnautov [3], Lovász [12], [13], and Payan [17] obtained a slightly better bound by considering a dominating set constructed by the following greedy algorithm: select the dominating set vertex by vertex in such a way that in every step we choose a vertex that dominates the maximum number of yet undominated vertices. (If there are more than one choices, we choose randomly.) They obtained

$$\gamma(G) \leq \frac{n}{\delta+1} H_{\delta+1}, \tag{2}$$

where $H_k = 1 + \frac{1}{2} + \ldots + \frac{1}{k}$ is the k^{th} harmonic number.

It can be deduced from the results in Alon [1] that these bounds are asymptotically optimal as $\delta \to \infty$. Nevertheless, the question of the optimal bound for the domination number is still interesting for small minimum degrees.

Another, related question is to study the optimal bound for regular graphs. It is obvious that for a d-regular graph G on n vertices we have

$$\gamma(G) \geq \frac{1}{d+1} \cdot n. \tag{3}$$

Denoting the class of d-regular graphs by $\mathcal{G}_d$, we may define the parameter $D(d)$ as

$$D(d) = \sup\left\{\frac{\gamma(G)}{|V(G)|} : G \in \mathcal{G}_d\right\} = \inf\left\{t : G \in \mathcal{G}_d, \gamma(G) \leq t|V(G)|\right\}. \tag{4}$$

As $D(d)$ gives an upper bound for the dominating number of d-regular graphs, it is a natural problem to find good lower/upper bounds for $D(d)$. We list the best available information about $D(d)$ for small d in Table 1 with references to the first publication of the relevant results.

For comparison, we note the best known upper bounds for the domination number of graphs of given small fixed minimum degree. Let G be a graph of order n. Ore [16] showed that if $\delta(G) \geq 1$, then $\gamma(G) \leq n/2$. McCuaig and Shepherd [14] showed that if $\delta(G) \geq 2$ (and G is not one of seven exceptional graphs), then $\gamma(G) \leq (2/5)n$. Reed [18] showed that if $\delta(G) \geq 3$, then $\gamma(G) \leq (3/8)n$. In the monograph [9] by Haynes, Hedetniemi and Slater there appeared the general conjecture that for $\delta(G) = k \geq 4$, $\gamma(G) \geq kn/(3k-1)$. The conjecture was proven in a sequence of papers, and in fact, for large values k, stronger inequalities are satisfied.

In Section 2 we prove a new upper bound for the dominating number for d-regular graphs. Our bound is significantly stronger than (1) in case of small $\delta = d$. While better bounds are known for $d < 6$ (see Table 1), our bound is the strongest known for any fixed $d \geq 6$.

In Section 3 we compute some exact values of the domatic number of certain Kneser graphs. Originally we hoped that odd graphs (i.e. Kneser graphs $K(2n+1, n)$, studied in [15]), which are $(n+1)$-regular, may provide improved lower bounds for $D(d)$, however, somewhat surprisingly, this is not the case for small d due to the existence of certain block designs.

d	$D(d)$
1	$\frac{1}{2}$
2	$\frac{1}{2}$
3	$\frac{3}{8}$ [18, 5]
4	$\leq \frac{4}{11}$ [11]
5	$\leq \frac{5}{14}$ [19]

TABLE 1. Upper bound for the domination number for small d

2. Upper bound for γ for d-regular graphs

Clark et al. [6] showed that the greedy algorithm proposed by Alon and Spencer in [2] can be used to give better bounds on the domination number of graphs with a fixed minimum degree. They also found further improvement for regular graphs. They showed that if $V(G) = n$ for a d-regular graph G, then

$$\gamma(G) \leq C_d n \quad \text{where} \quad C_d = 1 - \frac{d - 1 + \frac{1}{d^2+1}}{\prod_{j=1}^{d-1}(1 + \frac{d+1}{jd})}. \tag{5}$$

We prove a similar theorem with a strictly smaller constant. (Strictly speaking, Clark et al. [6] did not provide a proof to the improvement for d-regular graphs, just argued—correctly—that it is analogous to the proof in the minimum degree condition. If one accepts this claim, then he should our improvement as well, without a detailed proof. For the sake of completeness, however, we provide the proof.) Our main result is the following.

Theorem 2.1. *For a d-regular graph G on n vertices,*

$$\gamma(G) \leq B_d n \quad \textit{where} \quad B_d = 1 - \frac{d^2 - d + 1}{1 + d\prod_{j=1}^{d-1}\left(1 + \frac{d+1}{jd}\right)}. \tag{6}$$

Just from the formulas, it is not obvious that $B_d < C_d$. However, the proof easily implies this fact, and it is also easy to compare the constants for specific values of d, as they are both rational numbers. E.g. $C_6 = \frac{570971}{1690715} \approx 0.3377 > B_6 = \frac{483425}{1447649} \approx 0.3339$.

The theorem works for all $d \geq 1$, but, as mentioned in the introduction, for $d < 6$ there are stronger or equally strong results in the literature, listed in Table 1. Asymptotic analysis of the bounds are shown in Table 2.

$\frac{\ln d+1}{d}$	$-O\left(\frac{\ln d}{d^2}\right)$		
$\frac{\ln d+\gamma}{d}$	$-\frac{\ln d-\frac{3}{2}+\gamma}{d^2}$	$+\frac{12\ln d+12\gamma-31}{12d^3}$	$-O\left(\frac{\ln d}{d^4}\right)$
$\frac{\ln d+\gamma}{d}$	$-\frac{\frac{1}{2}(\ln d)^2+\gamma\ln d-\frac{3}{2}+\frac{\pi^2+6\gamma^2}{12}}{d^2}$	$+\frac{2(\ln d)^3+6\gamma(\ln d)^2+(6\gamma^2+\pi^2-18)\ln d+4\zeta(3)+\pi^2\gamma+2\gamma^3-7-18\gamma}{12d^3}$	$+O\left(\frac{(\ln d)^4}{d^4}\right)$
$\frac{\ln d+\gamma}{d}$	$-\frac{\frac{1}{2}(\ln d)^2+\gamma\ln d-\frac{3}{2}+\frac{\pi^2+6\gamma^2}{12}}{d^2}$	$+\frac{2(\ln d)^3+6\gamma(\ln d)^2+(6\gamma^2+\pi^2-30)\ln d+4\zeta(3)+\pi^2\gamma+2\gamma^3+5-30\gamma}{12d^3}$	$+O\left(\frac{(\ln d)^4}{d^4}\right)$

TABLE 2. For comparison, we include here asymptotic formulae for (1) [Alon [2]], (2) [Arnautov [3], Lovász [12], [13] and Payan [17]], (5) [Clark et al. [6]], and our bound (6), in this order, on the fraction $D(d)$ as function of d, as $d \to \infty$. We obtained these results with Maple. The first two bounds hold for graphs with minimum degree d as well. The Euler-Mascheroni constant is denoted by $\gamma = 0.5772...$, the Riemann zeta function in 3 is $\zeta(3)$. The successive improvements always happen in the coefficient of the next power of d.

2.1. **Proof of Theorem 2.1.** Let G be a graph and $T \subseteq V(G)$. One can build a dominating set of G starting from T and then using the greedy algorithm described above to add vertices to T, until a dominating set is reached. The following lemma is essentially an adapted form of the main argument in [6], keeping the same notation as far as possible for easy comparison.

Lemma 2.1.1. *Let G be a graph of minimum degree δ and a maximal $T \subseteq V(G)$ such that any two vertices in T are of distance at least 3. Let $n = |V(G)|$ and $t = |T|/n$. Suppose that the greedy algorithm builds a dominating set D starting from T. Then*

$$|D| \leq A_\delta n$$

where

$$A_\delta = 1 - \frac{\delta - 1 + t}{\prod_{j=1}^{\delta-1}\left(1 + \frac{\delta+1}{j\delta}\right)}.$$

Proof. Suppose that after a certain step in the greedy algorithm, the number of yet uncovered vertices is r, and the size of the built set is i. Each of the yet uncovered vertices have a closed neighborhood of cardinality at least $\delta + 1$, so their total cardinalities is at least $r(\delta + 1)$. The $n - i$ points that are not in the built set are on average in $r(\delta + 1)/(n - i)$ of these neighborhoods, so there exists a vertex that has at least $r(\delta + 1)/(n - i)$ uncovered neighbor. Note that this vertex may be covered. The arguments shows that in every new step, we decrease the number of uncovered vertices by at least $r(\delta + 1)/(n - i)$.

Define $g_0 = n - (\delta + 1)T$. We define the sequence g_ℓ, which is an upper bound on the number of not yet dominated vertices after ℓ greedy choices to extend T into a dominating set. For $\ell > 0$, we know from the averaging argument above that if there are r not yet dominated vertices after ℓ greedy steps, then after a greedy choice to dominate as much as possible, at most $r\left(1 - \frac{\delta+1}{n-\ell-|T|}\right)$ are not dominated, for any $\ell > 0$. Hence we set $g_{\ell+1} = \left\lfloor g_\ell\left(1 - \frac{\delta+1}{n-\ell-|T|}\right)\right\rfloor$. Our analogue of the $\gamma_6(G)$ bound for the domination number of G in [6] is $\gamma_6'(G) = |T| + \min\{\ell' : g_{\ell}' = 0\}$.

Observe that $g_{n-\delta} = 0$ and $g_0 = n - (\delta + 1)T > g_1 > g_2 > ... > g_{\gamma_6'(G)} = 0$. Define

$$R_{\ell+1} = \left\lceil g_\ell\left(1 - \frac{\delta + 1}{n - \ell - |T|}\right)\right\rceil$$

for $0 \leq \ell < \gamma_6'(G)$ and observe that in this range $g_\ell > g_{\ell+1} = g_\ell - R_{\ell+1}$, and hence $R_{\ell+1} \geq 1$. As T was maximal, we conclude that

$$\delta \geq R_1 \geq R_2 \geq .... \geq R_{\gamma_6'(G)} \geq 1.$$

For $2 \le j \le \delta+1$, set

$$n_j = |\{i : 1 \le i \le \gamma'_6(G) \text{ with } \ R_i = \delta+2-j\}|.$$

Clearly $\gamma'_6(G) = |T| + n_2 + \dots + n_{\delta+1}$. We are going to find a recursive formula for the n_i numbers.

For $1 \le \ell \le n_2$, we have $g_\ell = n-(\delta+1)|T| - \ell\delta$, and for any $0 \le n' \le n_k$ and $\ell = n_2 + \dots + n_{k-1} + n'$, we have

$$g_\ell = n - (\delta+1)|T| - \sum_{j=2}^{k-1}(\delta+2-j)n_j - n'(\delta+2-k).$$

It is clear that n_2 is the least integer such that $R_{n_2+1} \le \delta - 1$. Using the definition of R_{n_2+1} and the displayed formula above for g_{n_2}, this means

$$\left(n - (\delta+1)|T| - N_2\delta\right)\frac{\delta+1}{n-n_2-|T|} \le \delta - 1,$$

which is equivalent to

$$\frac{2n - |T|(\delta^2+\delta+2)}{\delta^2+1} \le n_2.$$

Hence $n_2 = \left\lceil \frac{2n-|T|(\delta^2+\delta+2)}{\delta^2+1} \right\rceil$.

With a similar argument, for $2 < k \le \delta+1$, we obtain that n_k is the least integer satisfying $R_{n_2+\dots+n_k+1} \le \delta + 2 - (k+1)$, which by a similar argument corresponds to

$$\left(n - (\delta+1)|T| - \sum_{j=2}^{k}(\delta+2-j)n_j\right)\frac{\delta+1}{n-|T|-(n_2+\dots+n_k)} =$$

$$g_{n_2+\dots+n_k}\frac{\delta+1}{n-|T|-(n_2+\dots+n_k)} \le \delta+2-(k+1).$$

Solving the inequality above for n_k, we obtain after some algebra

$$\frac{kn - (\delta^2+\delta+2)|T| - \sum_{j=2}^{k-1}(\delta^2+(2-j)\delta+k+1-j)n_j}{\delta^2+(2-k)\delta+1} \le n_k.$$

Using the Clark et al. [6] notation $a_j = (\delta+1)(\delta+1-j)$, we can write

$$n_k = \left\lceil \frac{kn-(a_1+k)|T| - \sum_{j=2}^{k-1}(a_j+k)n_j}{a_k+k} \right\rceil.$$

Consider now a recursion similar to that of n_k, but without taking ceilings. Set $s_2 = \frac{2n-(\delta^2+d+2)|T|}{\delta^2+1}$ and

$$s_k = \frac{kn-(a_1+k)|T| - \sum_{j=2}^{k-1}(a_j+k)s_j}{a_k+k}.$$

As the coefficients in the recursions for n_k and s_k are the same as those in Clark et al. [6] (see pages 15–16 in the paper on Clark's webpage), their inequality $n_2 + \dots + n_{\delta+1} \le s_2 + \dots + s_{\delta+1}$ holds, with the very same proof that they provided.

Following Clark et al. [6], define $u_j = s_j/n$, namely $u_2 = \frac{2}{\delta^2+1} - \frac{\delta^2+\delta+2}{\delta^2+1}\cdot t$ and $u_k = \frac{k-(a_1+k)t-\sum_{j=2}^{k-1}(a_j+k)u_j}{a_k+k}$. From the displayed formula above, for $k \ge 2$, we have the recurrence

$$k - (a_1+k)t = \sum_{j=2}^{k}(a_j+k)u_j.$$

Preparing for an Abel summation, set $S_1 = 0$, $S_k = \sum_{j=2}^{k} u_j$, $u_j = S_j - S_{j-1}$. Our goal is to compute $S_{\delta+1}$ and use the estimate $|T| + n\cdot S_{\delta+1} \ge \gamma'_6(G)$.

Start with

$$k-(a_1+k)t = \sum_{j=2}^{k}(a_j+k)(S_j - S_{j-1}) = S_k(a_k+k) + \sum_{j=2}^{k-1} S_j(\delta+1).$$

Define now $X_1 = 0$ and $X_k = \sum_{j=2}^{k} S_j$. Clearly $S_j = X_j - X_{j-1}$,

$$X_2 = \frac{2n-|T|(d^2+d+2)}{n(d^2+1)} \tag{7}$$

and the displayed formula above turns into

$$k-(a_1+k)t = (X_k - X_{k-1})(a_k+k) + (\delta+1)X_{k-1}$$

for $k \ge 3$. Replacing the a-terms with their definition, we obtain the inhomogeneous linear recurrence with polynomial coefficients

$$(\delta^2+(2-k)\delta+1)X_k = \delta(\delta-k+1)X_{k-1} + k - (\delta^2+\delta+k)t. \tag{8}$$

In (8) above, setting $k = \delta+1$, the coefficient of X_{k-1} vanishes, and therefore we obtain

$$X_{\delta+1} = 1-(\delta+1)t.$$

We solve the recurrence (8) with initial condition (7). Observe that $-\delta + (1-t)k$ is a particular solution to (8). The homogeneous recurrence corresponding to (8) is solved by $C\prod_{i=\delta-k+1}^{\delta-2}\left(1+\frac{\delta+1}{i\delta}\right)^{-1}$ with any C constant. Write

$$X_k = -\delta + (1-t)k + \frac{\delta - 2(1-t) + X_2}{\prod_{i=\delta-k+1}^{\delta-2}(1+\frac{\delta+1}{i\delta})}.$$

As this is the sum of a particular solution of (8) and a particular solution ($C = \delta - 2(1-t) + X_2$) of the homogeneous recurrence corresponding to

(8), this is a solution for (8), which meets the initial condition (7) for $k = 2$. Therefore it is a solution for $2 \le k \le \delta$. Finally

$$\gamma_6'(G)/n \le t + S_{\delta+1} = t + X_{\delta+1} - X_\delta$$

$$= \quad t + \Big(1 - (\delta+1)t\Big) - \left(-\delta + (1-t)\delta + \frac{\delta - 2(1-t) + X_2}{\prod_{i=1}^{\delta-2}(1 + \frac{\delta+1}{i\delta})}\right) = A_\delta.$$

□

To simplify the rest of the proof of Theorem, use the following notation:

$$P_d = \prod_{j=1}^{d-1}\left(1 + \frac{d+1}{jd}\right)$$

$$Q_d = 1 - \frac{d-1}{P_d}$$

$$R_d = \frac{1}{P_d}$$

In the next step, we will construct a small dominating set for G. Consider the the following algorithm. In the first step, we find a maximal $T \subseteq V(G)$ with the property that any two vertices in T are of distance at least 3. Note that $N(T)$, the open neighborhood of T is a dominating set. Let $t = |T|/n$, as in Lemma 2.1.1.

If $td < Q_d - tR_d$, then output $N(T)$. Otherwise start a greedy algorithm from T, and output the result.

Observe that $|N(T)| = tdn$. Also, due to Lemma 2.1.1, the greedy algorithm produces a dominating set of size at most $A_d n {=} (Q_d - tR_d)n$. So the constructed dominating set is the largest, if $td = Q_d - tR_d$. In that case

$$t = \frac{Q_d}{R_d + d} = \frac{P_d - d + 1}{1 + dP_d},$$

and the size of the constructed dominating set is at most

$$tdn = \left(1 - \frac{d^2 - d + 1}{1 + dP_d}\right) n = B_d n.$$

□□

3. Domination in Kneser graphs

As usual, set $[n] = \{1, 2, \ldots, n\}$. The Kneser graph $K(n,k)$ has vertex set $\binom{[n]}{k}$ and vertices $A, B \subseteq [n]$ are adjacent if they are disjoint. The cases $k = 1$ (when $K(n,k)$ is the complete graph on n vertices) and $\frac{n}{2} < k$ (when $K(n,k)$ is the empty graph) are trivial. Domination numbers of Kneser graphs have been studied in the manuscript of Hartman and West [8] and in the Master's Thesis of Gorodezky [7].

Recall that an $S(t,k,v)$ block design is a set of k-subsets (called blocks) of a v-set, where every t-subset of the v-set appears in precisely one block. Surprisingly, the lower bound (3) for the domination number is tight for the graphs $K(7,3)$ and $K(11,5)$, as the designs $S(2,3,7)$ and $S(4,5,11)$ dominate them (see [8], p. 34). So it turns out that the Kneser graphs $K(2k+1,k)$ form small k are not good examples for regular graphs of large dominating numbers. However we still may have some hope for larger odd graphs, since we do not have corresponding block designs there. It is worth noting that the Lovász Local Lemma does not seem to help at finding regular graphs of large dominating numbers.

Gorodezky [7] showed that for $n > k(k+1)$ it is true that $\gamma(K(n,k)) = \gamma_t(K(n,k)) = k+1$. Ivanco and Zelinka [10] showed that for $n \ge 6$, $\mathrm{dom}(K(n,2)) = \left\lfloor \frac{1}{3}\binom{n}{2} \right\rfloor$. We generalize this result below:

Theorem 3.1. *If $n \ge k(k+1)$ then* $\mathrm{dom}(K(n,k)) = \left\lfloor \frac{1}{k+1}\binom{n}{k} \right\rfloor$.

Proof. It is easy to see [7] that a collection of $k+1$ disjoint k-subsets of $[n]$ gives a (total) dominating set of $K(n,k)$.

Choose $q = \left\lceil \frac{1}{k+1}\binom{n}{k} \right\rceil$ and let $a_1 = \ldots = a_{q-1} = k+1$. and $a_q = \binom{n}{k} - \sum_{j<q} a_j$. Note that $1 \le a_q \le k+1$ and $a_j = k+1$ precisely when $j \le \left\lfloor \frac{1}{k+1}\binom{n}{k} \right\rfloor$. Also, $\sum a_j = \binom{n}{k}$. Baranyai's theorem [4] states that there are sets $\mathcal{Y}_1, \ldots, \mathcal{Y}_q \subseteq \binom{[n]}{k}$ such that $|\mathcal{Y}_i| = a_i$ and for all i and any $x, y \in [n]$ we have that the number of sets x and y is an element of in $\mathcal{Y}_i$ differs in at most 1. Thus, if $a_i = k+1$, then $\mathcal{Y}_i$ must be a (total) dominating set. □

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Applying Integer Programming to Enumerate Equilibrium States of a Multi-Link Inverted Pendulum: A Strange Binomial Coefficient Identity and its Proof

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Abstract

The task of counting certain (unstable) equilibrium states associated with an idealised 2D inverted pendulum system—some of which describe a particular type of random walk—is re-examined as an optimisation problem. In recovering a known general result, an unusual binomial coefficient identity is generated and requires a non-trivial proof. The complexity of the algorithm leading to state enumeration and identification is, as the number of links increases, discussed in the light of previous work. In addition, a similar type of identity is formulated (and proven) by appealing to some independent analysis by Carlitz from the 1970s, and generalised (computer-based) versions of both identities are included for completeness.