Math 501/Final Exam

Show all your work clearly, no credit otherwise. Look for more problems on the back.

1. True/False. Decide if the following statements are True or False. You will receive 4 points for deciding if the statement is T/F and 3 pts for a correct explanation.

   • If $A \subseteq \mathbb{R}$ is dense in $\mathbb{R}$, then $\mathbb{R} \setminus A$ is not dense in $\mathbb{R}$.

   • Suppose $A \subseteq \mathbb{R}$ is bounded and $\text{sup}(A)$ is not in $A$. Then, $A$ has at least one accumulation point.

   • If $A$ is uncountable and $B \subseteq A$ is countable, then $A \setminus B$ is countable.
• Let \( A \subseteq \mathbb{R} \) and suppose \( x \in A \). If \( x \) is not an interior point of \( A \), then \( x \) is a boundary point of \( A \).

• If \( A \subseteq \mathbb{R} \) is a nonempty, proper subset of \( \mathbb{R} \) and \( A \) is not closed, then \( A \) is open.

• If \( A_1, A_2, \ldots \) are compact sets, then \( \bigcap_{n=1}^{\infty} A_n \) is compact.

• There is a sequence of real numbers \( \{x_n\} \) such that every real number is the limit of some subsequence of \( \{x_n\} \).
• Every infinite subset of \( \mathbb{R} \) has an accumulation point.

• There is a continuous function \( f : [0, 1] \setminus \{ \frac{1}{2} \} \rightarrow \mathbb{R} \) which does not satisfy the intermediate value property.

• If \( E \subseteq \mathbb{R} \) is closed and \( f : E \rightarrow \mathbb{R} \) is continuous, then \( f(E) \) is closed.

• Every function \( f : \mathbb{N} \rightarrow \mathbb{R} \) is uniformly continuous.

• If \( A \subseteq [0, 1] \) is such that every point of \( A \) is an accumulation point of \( A \), then \( \overline{A} \) contains some interval.
2. (7 pts each) **Bare Denials.** Complete the following. I do **NOT** want that the definition is not true. I want you to negate the definition. There will be little partial credit for this section.

**Example:** Number $M$ is not an upperbound of set $A$ means that there is $x \in A$ such that $x > M$.

- Let $M \subseteq \mathbb{R}$. $M$ **does not** have the Bolzano-Weierstrass property means that

- Let $M \subseteq \mathbb{R}$ and $f : M \to \mathbb{R}$. $f$ **is not** uniformly continuous on $M$ means that

- Let $\{x_n\}$ be a sequence of real numbers. $\{x_n\}$ **is not** Cauchy means that

- Let $A \subseteq \mathbb{R}$. $A$ **is not** bounded means that
3. (8 pts each) Examples.

- Find a set $E \subseteq \mathbb{Q}$ such that the sup($E$) is an irrational number.

- Consider the sequence $\{x_n\}$ defined by $x_n = (-1)^n$. Show why this sequence is not Cauchy directly from the definition of Cauchy sequence.

- Find the sup, max, inf, min of the set $E = \mathbb{N} \cup M$ where $M$ is the set of positive rational numbers.

- Let $E = \mathbb{N} \cup M$ where $M$ is the set of positive rational numbers. Find the interior and the closure of $E$.

- Let $A = \mathbb{Q}$. Show that $A$ does not have the Heine-Borel property directly from the definition of Heine-Borel property (i.e. every open cover has a finite subcover.)
4. (10 pts) Use the $\epsilon - \delta$ definition of continuity to show that $f(x) = 3x^2 + 2x - 1$ is continuous at $x = 0$.

5. (10 pts) Show directly from the definition of Heine-Borel property that if $A, B$ have the Heine-Borel property, then so does $A \cup B$. 
6. (10 pts) Show that if $p$ is an accumulation point of $A \cup B$, then $p$ is an accumulation point of $A$ or $p$ is an accumulation point of $B$.

7. (10 pts) Clearly explain why the set of all infinite sequences of 0’s and 1’s is uncountable. Then, use this fact to explain why the Cantor set is uncountable.
8. (10 pts) Suppose $f : E \to \mathbb{R}$ is a Lipschitz function, i.e. there is $M > 0$ such that $|f(x) - f(y)| < M|x - y|$ for all $x, y \in E$. Show that $f$ is uniformly continuous on $E$.

9. (10 pts) Extra Credit for undergrads. Required for Grads. Suppose that $f$ is uniformly continuous on each of the compact sets $X_1, X_2, \ldots, X_n$. Show that $f$ is uniformly continuous on $X = \bigcup_{i=1}^n X_i$. 
