Math 501/Test 3

Show all your work clearly, no credit otherwise. Look for more problems on the back.

1. True/False. Decide if the following statements are True or False. You will receive 3 points for deciding if the statement is T/F and 3 pts for a correct explanation.

   • There is a compact set \( A \) and a noncompact set \( B \) such that \( A \cup B \) is compact.

   • Suppose \( f : A \to \mathbb{R} \) and \( x \in A \). If \( f \) is locally bounded at \( x \), then \( f \) is continuous at \( x \).

   • There is an uncountable closed subset of \([0,1]\) which contains no interval.
• There is a continuous function \( f : (0, 1) \to \mathbb{R} \) such that \( f((0, 1)) \) is not bounded.

• Let \( f : \mathbb{R} \to \mathbb{R} \) be such that \( |f| \) is continuous. Then, \( f \) is continuous.

• If \( A_1, A_2, \ldots \) is a sequence of compact sets, then \( \bigcup_{n=1}^{\infty} A_n \) is compact.

2. Bare Denials. Complete the following. I do NOT want that the definition is not true. I want you to negate the definition. There will be little partial credit for this section.

Example: Number \( M \) is not an upperbound of set \( A \) means that there is \( x \in A \) such that \( x > M \).
• Let $M \subseteq \mathbb{R}$. $M$ does not have the Heine-Borel property means that

• Let $M \subseteq \mathbb{R}$ and $p \in M$. $f : M \to \mathbb{R}$ is not continuous at $p$ means that

3. Examples.

• Let $A = [0, 2) \cup (2, 3]$. Show that $A$ does not have the Heine-Borel property directly from the definition.

• Let $A = \mathbb{N}$. Show why $A$ fails to satisfy the Bolzano-Weierstrass property directly from the definition.
4. Use the $\epsilon - \delta$ definition of continuity to show that $f(x) = 3x^2 + 2x - 1$ is continuous at $x = 1$.

5. Show directly from the definition of Heine-Borel property that if $E$ is compact and $F \subseteq E$ is closed, then $F$ has the Heine-Borel property.
6. Let \( f, g : A \rightarrow \mathbb{R} \) be functions. Use the \( \epsilon - \delta \) definition of continuity to show that if \( f \) is continuous at \( p \) and \( g \) is continuous at \( p \), then \( f + g \) is continuous at \( p \).

7. Let \( A \) be a countable subset of \( \mathbb{R} \). Show that there is a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) which is continuous at every \( x \in \mathbb{R} \setminus A \) and discontinuous at every \( x \in A \).
8. **Extra Credit for undergrads. Required for Grads.** Show that there is a function $f : \mathbb{R} \to \mathbb{R}$ which is not locally bounded at any point of $\mathbb{R}$. 