Math 502/Final Exam

Show all your work clearly, no credit otherwise. Look for more problems on the back.

1. True/False. Decide if the following statements are True or False. You will receive 4 points for deciding if the statement is T/F and 4 pts for a correct explanation.

   • If each of $A_1, A_2, \ldots$ is open and dense in $\mathbb{R}$, then $\bigcap_{n=1}^{\infty} A_n$ is dense in $\mathbb{R}$.

   • Every closed nowhere dense set is countable.

   • If $f : [0, 1] \to \mathbb{R}$ has a derivative at every point, then $f'$ is Riemann integrable.
• If \( f : \mathbb{R} \to \mathbb{R} \) is a function which is neither increasing nor decreasing, then \( f \) has a derivative zero at some point.

• There is a bounded function which is not Riemann integrable.

• The characteristic function of the middle 1/3 Cantor set is Riemann integrable.

• Suppose each of \( f_n : [0, 1] \to \mathbb{R} \) is a differentiable function and \( \{f_n\} \) converges uniformly to the zero function. Then, \( \{f_n'\} \) must converge pointwise to the zero function.
• There is a sequence of continuous functions \( \{f_n\} \) defined on \([0, 1]\) which converges pointwise to the saltNpepe function.

2. Explain why there is not a sequence of open sets whose intersection is precisely the set of rational numbers.

3. Give an example of a \textbf{continuous} function \( f : \mathbb{R} \to \mathbb{R} \) such that \( f'_+(0) = \infty \) and \( f'_-(0) = 2 \).
4. Suppose that \( f \) is differentiable on \([0, \infty)\) and that

\[
\lim_{x \to \infty} f'(x) = C.
\]

What is

\[
\lim_{x \to \infty} [f(x + a) - f(x)]?
\]

5. Prove that the countable union of measure zero sets has measure zero.
7. (15 pts) Find the set of points of convergence of the power series \( \sum_{n=1}^{\infty} \frac{x^n}{n} \). You may use any tests from the lecture, homework problem, etc. What is the radius of convergence?
8. (15 pts) Let

\[ y(x) = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \ldots. \]

Explain why \( y(x) \) can be term by term differentiated. Then, show that \( y' = 2xy \).
9. (15 pts) Suppose \( \{f_n\} \) is a sequence of continuous function defined on \([0,1]\) which converges uniformly to a function \( f \). Explain why \( f \) is integrable. Then, show that \( \{F_n\} \) converges uniformly to \( F \) where \( F_n(x) = \int_0^x f_n(t) \, dt \) and \( F(x) = \int_0^x f(t) \, dt \).

10. (15 pts) Prove that if each of \( \{f_n\} \) and \( \{g_n\} \) converge uniformly to functions \( f \) and \( g \), respectively, then \( \{f_n + g_n\} \) converge uniformly to \( f + g \).
11. (15 pts) Find the set of points of convergence of the power series \( \sum_{n=1}^{\infty} \frac{x^n}{n} \). You may use any tests from the lecture, homework problem, etc. What is the radius of convergence?
12. (15 pts) Let 

\[ y(x) = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \ldots \] 

Explain why \( y(x) \) can be term by term differentiated. Then, show that \( y' = 2xy \).
13. (15 pts) Suppose \( \{f_n\} \) is a sequence of continuous function defined on \([0,1]\) which converges uniformly to a function \( f \). Explain why \( f \) is integrable. Then, show that \( \{F_n\} \) converges uniformly to \( F \) where
\[
F_n(x) = \int_0^x f_n(t)\,dt \quad \text{and} \quad F(x) = \int_0^x f(t)\,dt.
\]
(10 pts) **Bonus for undergrads. Required for grads.**

14. Let \( \{f_n\} \) be a sequence of continuous functions defined on \([0, 1]\) which converge pointwise to some function \(f\). Show that there is some interval \([a, b] \subseteq [0, 1]\) and number \(M\) so that \(|f_n(x)| < M\) for all \(n\) and all \(x \in [a, b]\).