Math 502/Test 1

SHOW ALL WORK!!! NO CREDIT OTHERWISE! LOOK OR MORE PROBLEMS ON THE BACK!

1. Decide if the following statements are True or False. You will get 3 points for deciding if the statement is T/F and 3 points for a correct explanation.
   - If each of $A_1, A_2, \ldots$ is dense in $\mathbb{R}$, then $\bigcap_{n=1}^{\infty} A_i$ is dense in $\mathbb{R}$.
   - A set which has no interior points is nowhere dense.
   - Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ has derivatives of all order. If $f^{(n)}(0) = 0$ for all $n = 0, 1, 2, \ldots$, then $f$ is identically zero.
• There is a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that \{f'(x) : x \in \mathbb{R}\} is an unbounded set.

• Suppose $f : [-1, 1] \to \mathbb{R}$ is continuous. Furthermore, assume that $f'(x) < 0$ for all $-1 \leq x < 0$ and $f'(x) > 0$ for all $0 < x \leq 1$. Then $f'(0) = 0$.

2. Explain why $\mathbb{R} \setminus \mathbb{Q}$ is not meager.

3. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $f'_+(0) = \infty$ and $f'_-(0) = 2$. 
4. Show that the following function is continuous at 0 but not differentiable at any point.

\[ f(x) = 0 \text{ if } x \text{ is rational and } f(x) = x \text{ if } x \text{ is irrational.} \]

5. Exhibit the Taylor’s polynomial about \( x = 0 \) of degree \( n \) for the function \( f(x) = \cos(x) \). Find \( n \) so that \( |R_n(x)| < 10^{-6} \) for all \( x \in [0, 3] \).

6. Prove that if \( f \) is differentiable at \( x \), then \( f \) is continuous at \( x \).
7. State the Mean Value Theorem and use it to prove that if \( f'(x) > 0 \) for all \( x \in (0, 1) \) then \( f \) is strictly increasing on \((0, 1)\).

8. Suppose \( f : [0, \infty) \to \mathbb{R} \) has the property that \( f'(x) > 0 \) for all \( x \geq 0 \). Show that if \( \lim_{x \to \infty} f(x) \) is some finite number, then there is an increasing sequence \( \{x_n\} \) such that \( \lim_{n \to \infty} f'(x_n) = 0 \).
9. (Required for grad students. Bonus for undergrads.) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at every point. Moreover, assume that $\lim_{x \to \infty} f'(x) = 0$. Does this imply that $\lim_{x \to \infty} f(x)$ exists and is finite?