1. Determine whether the following statements are true or false. You will get 3 points for determining whether each statement is T/F and 4 points for a correct explanation.

- There is a nonabelian group of order 30.
- Up to isomorphism there are only 4 abelian groups of order 36.
- Every group of order 341 is abelian.
- There is an ideal of $\mathbb{Q}[x]$ which is not principal.
- There is a noncommutative finite ring without an identity.
• There is a finite integral domain $D$ and $x \in D$ with $x \neq 0$ such that $x$ has no inverse in $D$.

• $GF(81)$ has a subfield of order 27.

• $f(x) = x^3 + 2x + 1$ has no multiple zeros in any extension of $\mathbb{Z}_3$.

• If $f$ is a ring homomorphism from $\mathbb{R}$ onto a ring $S$, then $S$ is a field.

2. Show that there are two Abelian groups of order 108 that have exactly 13 subgroups of order 3.
3. Prove that if $H$ is a normal subgroup of $G$ and $K$ is any subgroup of $G$, then $HK$ is a subgroup of $G$.

4. Find the characteristic of the ring $\mathbb{Z}_5 \oplus \mathbb{Z}_7$.

5. Find all zero divisors of $\mathbb{Z} \oplus \mathbb{Q} \oplus \mathbb{Z}$. 
6. Prove that \( A = \{(3x, y) : x, y \in \mathbb{Z}\} \) is a maximal ideal of \( \mathbb{Z} \oplus \mathbb{Z} \).

7. Show why ring 5\( \mathbb{Z} \) is not isomorphic to 25\( \mathbb{Z} \).

8. Let \( f(x) = 5x^4 + 3x^3 + 1 \) and \( g(x) = 3x^2 + 2x + 1 \) in \( \mathbb{Z}_7[x] \). Determine the quotient and remainder upon dividing \( f(x) \) by \( g(x) \).
9. Determine whether the following polynomial is irreducible over $\mathbb{Q}$.
   \[ p(x) = x^5 + 9x^4 + 12x^2 + 6. \]

10. Find all monic polynomials of degree 2 over $\mathbb{Z}_3$.

11. Consider $f(x) = x^3 + x + 1 \in \mathbb{Z}_2[x]$. Find a splitting field of $f$ over $\mathbb{Z}_2$.
    Then, factor $f(x)$ into linear factors over this splitting field.
12. Consider \( f(x) = x^3 + x + 1 \in \mathbb{Z}_2[x] \). Find a generator for the multiplicative group of the field \( \mathbb{Z}_2[x]/<f(x)> \).