1. Determine whether the following statements are true or false. You will get 4 points for determining correctly whether the statement is true or false and 4 points for a correct explanation.

(a) Let $R$ be a finite commutative ring with identity and $I \subseteq R$ be a prime ideal of $R$. Then, $I$ is a maximal ideal of $R$.

(b) If $F$ is a field, then every ideal of $F[x]$ is a principal ideal domain.

(c) There is an infinite field $F$ which contains an isomorphic copy of $\mathbb{Z}_2$ as well as $\mathbb{Q}$.

(d) Let $I = \{ f(x) \in \mathbb{Z}[x] : f(0) = 0 \}$. Then, $I$ is a prime ideal of $\mathbb{Z}[x]$. 
(e) There is a ring isomorphism from $\mathbb{R}[x]/ < x^2 + 1 >$ to $\mathbb{C}$, the set of complex numbers.

2. Give an example of each of the following. If it does not exist, state why. Each one is worth 8 points.

(a) A finite noncommutative ring without an identity.

(b) An integral domain which is not a principal ideal domain.

(c) A ring $R$ and a polynomial $p(x) \in R[x]$ of degree 2 such that $p(x)$ has more than 2 roots over $R$.

(d) A ring homomorphism from $\mathbb{Z}$ onto $5\mathbb{Z}$. 
(e) An infinite integral domain which has characteristic 3.

3. Consider $R = \mathbb{Z}_2[x]$ and $I = \langle x^2 + x + 1 \rangle$. List all the distinct cosets of $R/I$. Then, find the multiplicative inverse of $[x + 1]$ in $R/I$.

4. Show that the only ideals of a field $F$ are $\{0\}$ and $F$ itself.
5. **Required for Graduate Students. Bonus for Undergrads.** Let $F$ be a finite field with $n$ elements. Show that $x^{n-1} = 1$ for all nonzero $x \in F$.

6. **Bonus for everyone.** Suppose $a, b$ belong to a field $F$ which has order $2^n$ for some odd integer $n$ and that $a^2 + ab + b^2 = 0$. Prove that $a = 0$ and $b = 0$. 