1. Determine whether the following statements are true or false. You will get 4 points for determining correctly whether the statement is true or false and 4 points for a correct explanation.

(a) Suppose $F$ is a finite field of cardinality $k$ and $f \in F[x]$ is an irreducible polynomial of degree $n$. Then, $F[x]/\langle f(x) \rangle$ has $k^n$ elements.

(b) There are at least four distinct polynomials of degree 2 which are irreducible over $\mathbb{Z}_3$.

(c) Suppose $f \in \mathbb{Z}[x]$ and $f(x) \mod 2$ is reducible over $\mathbb{Z}_2$. Then, $f$ is reducible over $\mathbb{Q}$.

(d) $GF(2^8)$ has a subfield consisting of 32 elements.
(e) \( \mathbb{Q}(\sqrt{2}) \) and \( \mathbb{Q}(-\sqrt{2}) \) are isomorphic.

2. Determine whether the following polynomial is irreducible over \( \mathbb{Q} \).
\[ f(x) = x^6 + 3x^5 + 6x^2 + 1. \]

3. Show that \( x^4 + x + 1 \) over \( \mathbb{Z}_2 \) does not have any multiple zero in any extension of \( \mathbb{Z}_2 \).
4. Show that $x^2 + x + 4$ is irreducible over $\mathbb{Z}_{11}$. Is it irreducible over $\mathbb{Q}$?

5. Find the splitting field of $f(x) = x^3 + 1 = (x + 1)(x^2 + x + 1)$ over $\mathbb{Z}_2$. Then, write $f(x)$ as the product of linear factors over the splitting field.
6. **Required for Graduate Students. Bonus for Undergrads.** Show that \([x]\) is not a generator of the cyclic group \((\mathbb{Z}_3[x]/ < x^3 + 2x + 2 >)^*\). Here, \((\mathbb{Z}_3[x]/ < x^3 + 2x + 2 >)^*\) denotes all the nonzero elements of the field \(\mathbb{Z}_3[x]/ < x^3 + 2x + 2 >\) under multiplication.

7. **Bonus for everyone.** Prove that for every prime \(p\),
\[
x^{p-1} - 1 = (x - 1)(x - 2) \ldots [x - (p - 1)]
\]
in \(\mathbb{Z}_p[x]\).