Markov Chains

- PROPERTIES
- REGULAR MARKOV CHAINS
- ABSORBING MARKOV CHAINS

Properties of Markov Chains

- Introduction
- Transition & State Matrices
- Powers of Matrices
- Applications

Andrei Markov 1856 -- 1922

Examples of Stochastic Processes

1) Stock Market UP DOWN UNCHANGED

2) Brand Loyalty:

Stay with brand A

Switch to brand A

Switch away from brand A

3) Brownian Motion

Product Loyalty

A marketing campaign has the effect that:

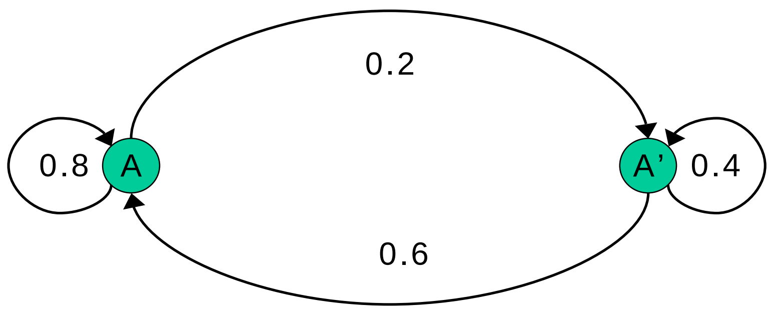
80 % of consumers who use brand A stay with it (so 20% switch away from it)

60 % consumers who use other brands switch to brand A

What happens in the long run?

Problem: FEEDBACK!

State transition diagram



		A	A'	next state
current state	A	0.8	0.2	
	A'	0.6	0.4	

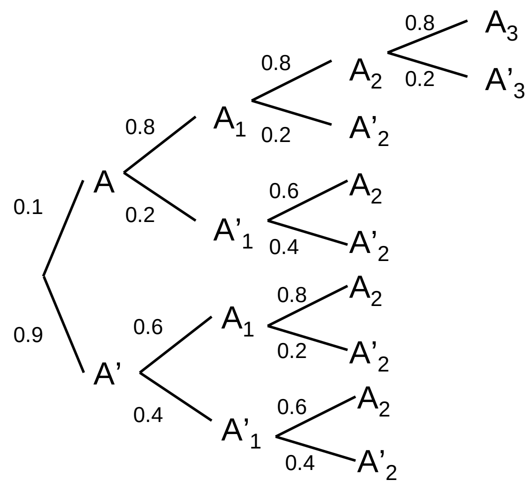
To determine what happens, we need to know the current state, that is the percentage of consumers buying brand A

Before the marketing campaign, brand A had a 10% market share:

	A	A'
S_0=	[0.1	0.9]

Initial State Probability Matrix

Probability that a randomly picked consumer buys brand A , does not buy brand A



Probabilities

Probability of switching to A after one week of marketing:

$P(A_1)$ $= P(A_0 \text{ and } A_1)+P(A'_0 \text{ and } A_1)$

$= P(A_0)P(A_1|A_0)+P(A'_0)P(A_1|A'_0)$

$= 0.1*0.8 \quad + 0.9*0.6 = 0.62$

First State Matrix: A A'

$S_1 = [\ 0.62 \quad 0.38\ ]$

$$S_1 = S_0 P = [\ 0.1 \quad 0.9\ ]\begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$

If the marketing campaign keeps having the same effect week after week, then the same matrix P applies each week:

After week 1: $S_1 = [\ 0.62 \quad 0.38\ ]$

After week 2: $S_2 = S_1P = (S_0P)P = S_0P^2$

$S_2 = [\ 0.724 \quad 0.276\ ]$

$$P^2 = PP = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.76 & 0.24 \\ 0.72 & 0.28 \end{bmatrix}$$

Markov Chains or Processes

- Sequence of trial with a constant transition matrix P
- No memory (P does not change, we do not know whether or how many times P has already been applied)

A Markov process has n states if there are n possible outcomes. In this case each state matrix has n entries, that is each state matrix is a 1 x n matrix.

The k-th state matrix is the result of applying the transition matrix P k times to an initial matrix S_0.

$S_k = [\ s_{k1}\ \ s_{k2}\ \ s_{k3} \dots s_{kn}\ ]$ where s_{ki} is the proportion of the population in state i after k trials.

The transition matrix P is a constant square matrix (n x n if there are n states) where the (i,j)-th element (i-th row, j-th column) gives the probability of transition from state i to state j.

Thus all entries are between 0 and 1,

$$0 \leq p_{ij} \leq 1$$

and all rows add up to 1,

$$p_{11}+p_{12}+\dots+p_{1n}=1$$

$$S_1 = S_0P$$

$$S_2 = S_1P = S_0PP = S_0P^2$$

$$S_3 = S_2P = S_0P^2P = S_0P^3$$

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$$S_k = S_{k-1}P = S_0P^k$$

Long Run Behavior of P

$$P^2 = \begin{bmatrix} 0.76 & 0.24 \\ 0.72 & 0.28 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0.752 & 0.248 \\ 0.744 & 0.256 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 0.7504 & 0.2496 \\ 0.7488 & 0.2512 \end{bmatrix}$$

$$P^{16} = \begin{bmatrix} 0.7500000001 & 0.2499999999 \\ 0.7500000001 & 0.2499999999 \end{bmatrix}$$

$$P^{\infty} = \begin{bmatrix} 0.75 & 0.25 \\ 0.75 & 0.25 \end{bmatrix}$$

Long Run Behavior of S

$$S_4 = [\,0.74896 \quad 0.25104\,]$$
$$S_{16} = [\,0.75000 \quad 0.25000\,]$$

Running the marketing campaign for a long time is in-effective, after 4 weeks already 74.896% are buying brand A. In the next 12 weeks only 0.104% more switch to brand A.

Note that these numbers are overly accurate, the model can NOT be that good.

Question

Does P^∞ always exist? NO!

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad P^{2k} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad P^{2k+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Better question: When does P^∞ exist?

Regular Markov Chains

- STATIONARY MATRICES
- REGULAR MARKOV CHAINS
- APPLICATIONS
- APPROXIMATIONS

Recall: Brand Switch Example

A: 80% stay with Brand A
20% switch to another Brand (A')

A': 60% Move to A (from A')
40% do not move (still use another brand)

$$P = \begin{array}{c} \\ A \\ A' \end{array}\begin{array}{c} \begin{array}{cc} A & A' \end{array} \\ \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

Initial Market Share

A : 10%
A’: 90%
S_0 = [0.1 0.9]
S_1 = S_0P = [0.62 0.38]
S_2 = S_0P^2 = [0.724 0.276]
S_4 = [0.74896 0.25104]
S_{10} = [0.7499 0.2501]
S_{20} = [0.749999 0.250001]

In the Long Run

S = [0.75 0.25] Stationary State Matrix

$$SP = \begin{bmatrix} 0.75 & 0.25 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.75 & 0.25 \end{bmatrix}$$

Stationary = Nothing Changes

Stationary State Matrix

The state matrix $S=[s_1\ s_2\ \ldots\ s_n]$ is a stationary state matrix for a Markov chain with transition matrix P if

$$SP = S$$

Where $s_i \geq 0$ and $s_1+s_2+\ldots+s_n = 1$.

Questions:

- Are stationary state matrices unique?
- Are stationary state matrices attractive?
- What is attracted?
- Can we tell by looking at P?

Regular Matrices
Regular Markov Chains

A transition matrix P is regular if some power of P has only positive (strictly greater than zero) entries.

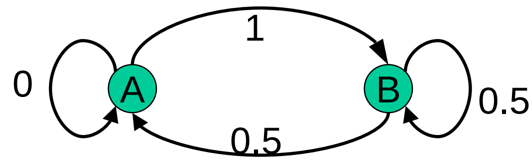
A regular Markov Chain is one that has a regular transition matrix P.

Examples of regular matrices

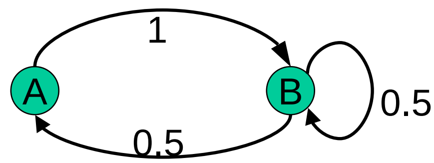
$$P = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.5 \end{bmatrix} \quad P^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{bmatrix}$$

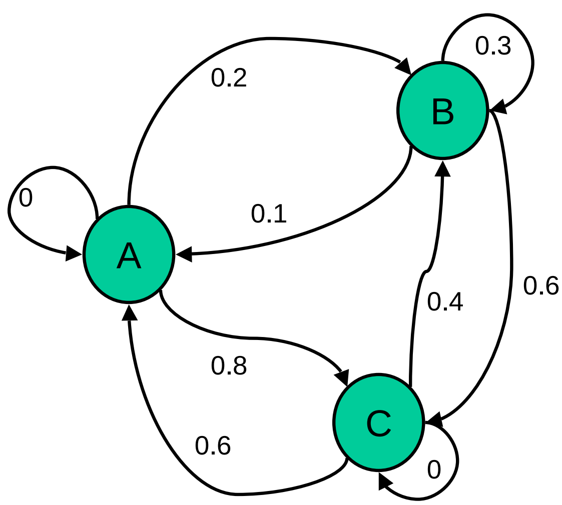
$$P = \begin{bmatrix} 0 & 0.2 & 0.8 \\ 0.1 & 0.3 & 0.6 \\ 0.6 & 0.4 & 0 \end{bmatrix} \quad P^2 = \begin{bmatrix} 0.5 & 0.38 & 0.12 \\ 0.39 & 0.35 & 0.26 \\ 0.4 & 0.24 & 0.72 \end{bmatrix}$$

Examples Of
Regular Markov Chains



We may leave out loops of zero probability





Theorem 1

Let P be a transition matrix for a regular Markov Chain

(A) There is a unique stationary matrix S, solution of

$$SP=S$$

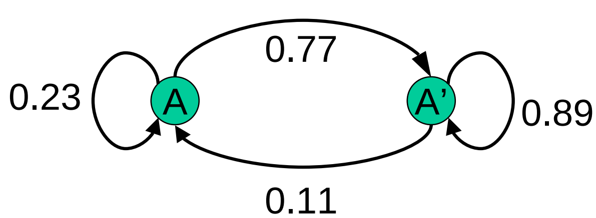
(B) Given any initial state S_0 the state matrices S_k approach the stationary matrix S

(C) The matrices P^k approach a limiting matrix $\overline{P}$,where each row of $\overline{P}$ is equal to the stationary matrix S.

Example 2 (Insurance Statistics)

23% of drivers involved in an accident are involved in an accident in the following year (A)

11% of drivers not involved in an accident are involved in an accident in the following year (A')



Example 2 (continued)

If 5% of all drivers had an accident one year, what is the probability that a driver, picked at random, has an accident in the following year?

		Next year	
		A	A'
This	A	0.23	0.77
year	A'	0.11	0.89

$$P = \begin{bmatrix} 0.23 & 0.77 \\ 0.11 & 0.89 \end{bmatrix}$$

S_0 = [0.05 0.95]
$S_1 = S_0P$ = [0.116 0.884], Prob(accident)=0.116

Example 2 (continued)

What about the long run behavior? What percentage of drivers will have an accident in a given year?

Since all entries in P are greater than 0, this is a regular Markov Chain and thus has a steady state:

$$P^2 = \begin{bmatrix} 0.1376 & 0.8624 \\ 0.1232 & 0.8768 \end{bmatrix} \quad P^3 = \begin{bmatrix} 0.126512 & 0.873488 \\ 0.124784 & 0.875216 \end{bmatrix}$$

$$P^{20} = \begin{bmatrix} 0.125 & 0.875 \\ 0.125 & 0.875 \end{bmatrix}$$

12.5% of drivers will have an accident.

Exact solution

By Theorem 1 part (A): Solve the equation S=SP

$$S = [s_1 \quad s_2], \text{ where } s_1 + s_2 = 1, \; P = \begin{bmatrix} 0.23 & 0.77 \\ 0.11 & 0.89 \end{bmatrix}$$

$$SP = [0.23s_1 + 0.11s_2 \quad 0.77s_1 + 0.89s_2]$$

$$S = SP \Leftrightarrow \begin{matrix} s_1 = 0.23s_1 + 0.11s_2 \\ s_2 = 0.77s_1 + 0.89s_2 \end{matrix} \text{ with } s_2 = 1 - s_1$$

$$s_1 = 0.23s_1 + 0.11(1 - s_1) \Leftrightarrow 0.88s_1 = 0.11 \Leftrightarrow s_1 = 0.125$$

$$\Rightarrow s_2 = 1 - s_1 = 1 - 0.125 = 0.875$$

Absorbing Markov Chains

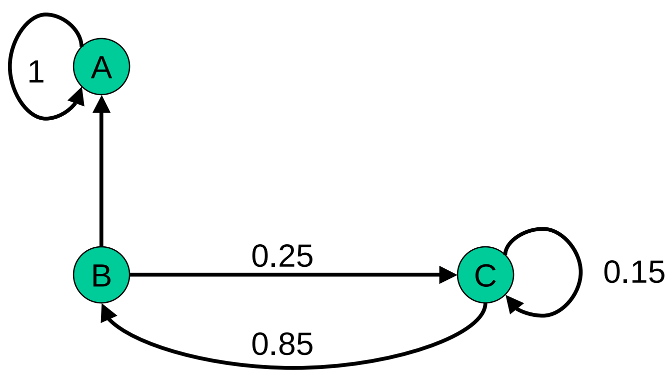
- Absorbing States and Chains
- Standard Form
- Limiting Matrix
- Approximations

Definition

A state is absorbing if, once the state is entered, it is impossible to leave it.

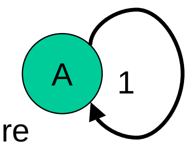
- No arrow leaving the state to other state
- One arrow returning to state itself with 1

Example 1



Observation

The number on an entering arrow gives the probability of entering that state from the state where the arrow started.



If you are at A then you stay there with probability 1, that is “for sure”.
Since all arrows leaving add up to 1, there is no other arrow leaving A.

Another Example

	To	A	B	C
	A	**1**	**0**	**0**
From	B	**0.75**	**0**	**0.25**
	C	**0**	**0.85**	**0.15**

Probability:
From A to A is 1
From A to B or C it is 0
A is absorbing

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0.75 & 0 & 0.25 \\ 0 & 0.85 & 0.15 \end{bmatrix}$$

Recall: Rows add to 1
Absorbing states have a **1** and **0’s** in their corresponding row.

Theorem 1:

A state in a Markov Chain is **absorbing** if and only if the row corresponding to the state has a **1 on the main diagonal** and **0's everywhere else**.

Absorbing versus Stationary

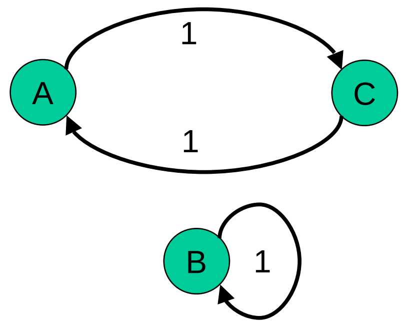
Absorbing does **NOT** imply that the states approach a stationary state. Recall a previous example:

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{2n+1} = P, \quad P^{2n} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So an absorbing state does not mean the matrix powers approach a limiting matrix

What went wrong?



B is absorbing, but A and C keep “flipping”

Definition:

A Markov Chain is an **absorbing chain** if

1) There is at least one absorbing state

2) It is possible to go from each non absorbing state to at least one absorbing state in at a finite number of steps

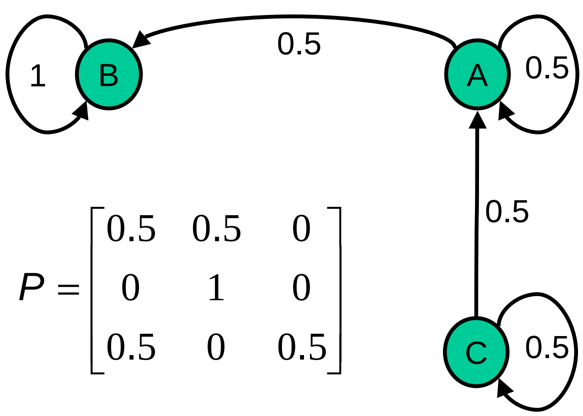
Another Definition:

A transition matrix for an absorbing Markov Chain is in **standard form** if the rows and columns are labeled so that all the absorbing states precede all the non absorbing states:

$$\begin{array}{c} \\ Abs. \\ NA. \end{array} \begin{array}{c} \begin{array}{cc} Abs. & NA. \end{array} \\ \left[\begin{array}{c|c} \mathrm{I} & 0 \\ \hline R & Q \end{array}\right] \end{array}$$

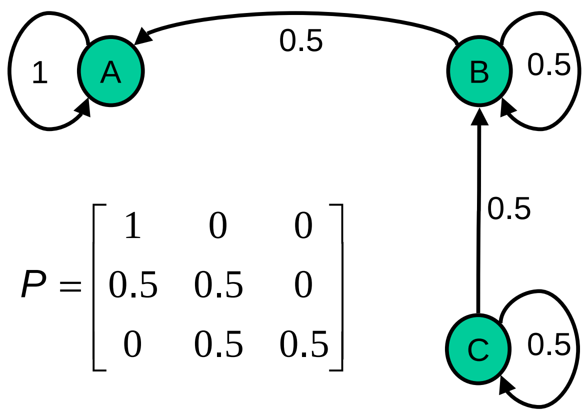
I is the identity matrix

Example:



$$P = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

Example:(contd.)



$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

Limiting Matrix:

If P is the matrix of an absorbing Markov Chain and P is in standard form, then there is a limiting matrix

$\overline{P}$ such that $P^k \to \overline{P}$ as k increases, where

$\overline{P} = \left[\begin{array}{c:c} \mathrm{I} & 0 \\ \hdashline FR & 0 \end{array}\right]$ and $F = (\mathrm{I} - \mathrm{Q})^{-1}$ Fundamental Matrix

$$\begin{array}{cc} & \begin{array}{cc} Abs. & NA. \end{array} \\ \begin{array}{c} Abs. \\ NA. \end{array} & \left[\begin{array}{c:c} \mathrm{I} & 0 \\ \hdashline R & Q \end{array}\right] \end{array}$$

More Examples:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix} \qquad R = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix},\ Q = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{bmatrix},$$

$$(\mathrm{I} - Q)^{-1} = \begin{bmatrix} 0.5 & 0 \\ -0.5 & 0.5 \end{bmatrix}^{-1}$$

$$F = \begin{bmatrix} 0.5 & 0 \\ -0.5 & 0.5 \end{bmatrix}^{-1} = \frac{1}{(0.5)(0.5)} \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{bmatrix}$$

$$4\begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix},\ \mathrm{FR} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \overline{P} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Without the Theorem:

$$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0.75 & 0.25 & 0 \\ 0.75 & 0 & 0.25 \end{bmatrix},\ P^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0.9375 & 0.0625 & 0 \\ 0.9375 & 0 & 0.0625 \end{bmatrix}$$

$$P^{16} = \begin{bmatrix} 1 & 0 & 0 \\ 0.9999847412 & 0.00001525878906 & 0 \\ 0.9999847412 & 0 & 0.00001525878906 \end{bmatrix}$$