Cheating will be punished. Show your work logically and write neatly. Just guessing answer won’t get any credit. **Write your work within the space given below, which will be possible!**

**Problem 1.** Define a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$T(x_1, x_2) = (x_1 - 2x_2, x_1 + 3x_2, -3x_1 + 2x_2).$$

(a) (10pt) Find the standard matrix $A$ for $T$.

(b) (10pt) Find $x = (x_1, x_2)$ such that $T(x) = (8, -7, -12)$. 
Problem 2. Let

\[
A = \begin{bmatrix}
1 & -2 & 0 \\
-3 & 4 & 1 \\
2 & 4 & -3
\end{bmatrix}.
\]

(a) (15pt) Find the inverse of \( A \).

(b) (5pt) Double check your answer by multiplying \( A \) by its inverse to get an identity matrix.
Problem 3. Assume that matrix $A$ and $B$ are row equivalent.

$$A = \begin{bmatrix}
1 & 4 & -3 & -1 & 9 \\
-2 & -6 & 6 & -1 & -10 \\
-3 & -6 & 9 & -6 & -3 \\
3 & 4 & -9 & 9 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 & -3 & 5 & -7 \\
0 & 2 & 0 & -3 & 8 \\
0 & 0 & 0 & 0 & 5 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

(a) (15pt) Find bases for $\text{Col}(A)$, $\text{Row}(A)$, and $\text{Nul}(A)$.

(b) (5pt) Find also $\text{rank}(A)$ and dimension of $\text{Nul}(A)$ and then show it satisfies the rank theorem.
Problem 4. Let \( A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \). Answer the following.

(a) (5pt) Find all eigenvalues of \( A \) by finding the characteristic equation of \( A \). Include the multiplicity of each eigenvalue.

(b) (10pt) Find a basis for each corresponding eigenspace of \( A \).

(c) (5pt) Determine whether \( A \) is diagonalizable.
Problem 5. Let \( A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \).

(a) (5pt) Determine whether \( A \) and \( B \) are diagonalizable. Justify your answer to get the full credit.

(b) (15pt) If \( A \) or \( B \) is diagonalizable, then find \( P \) and \( D \) so that \( PDP^{-1} \) is \( A \) or \( B \).

Problem 6. Let \( \{x_1, x_2\} \) be a basis for \( W = \text{span}\{x_1, x_2\} \), where

\[
x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix},
\]

(a) (10pt) Find an orthogonal basis \( \{u_1, u_2\} \) for \( W \).
Problem 7. (a) (15pt) Find an orthogonal basis for the column space of $A$, where

$$A = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
1 & 2 & 1
\end{bmatrix}.$$  

(b) (5pt) Double-check whether your basis is really orthogonal.

(b) (10pt) Let $y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Write $\hat{y} = \text{Proj}_W y$ in term of your new orthogonal basis obtained in (a).
Problem 8. Let

\[ A = \begin{bmatrix}
1 & -1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
1 & 0 & -1 & 0 \\
1 & 0 & -1 & 0 \\
1 & 0 & 0 & -1 \\
1 & 0 & 0 & -1 \\
\end{bmatrix}, \quad b = \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
1 \\
\end{bmatrix}. \]

(a) (5pt) Show that the equation \( Ax = b \) is inconsistent by inspection.

(b) (15pt) Find all least-squares solutions of the equation \( Ax = b \).

(c) (10pt) Compute the least-squares error associated with the least-squares solutions found in (a).
Problem 9. (a) (5pt) Give a definition of the similarity of two $n$ by $n$ matrices $A, B$.

(b) (5pt) Then prove that if two $n$ by $n$ matrices $A, B$ are similar, they have the same characteristic polynomial.

Problem 10. (20pt) Circle True or False for each statement. If true give a brief justification and if false give a counterexample, to get full credit. Each is worth 4 points.

(a) True or False : Every vector space with dimension at least 2 has an orthogonal basis.

(b) True or False : If the least-squares error of a matrix equation $Ax = b$ is zero, then $Ax = b$ is inconsistent.

(c) True or False : If $U$ is an $n$ by $n$ matrix such that $U^T U = I_n$, where $I_n$ is an identity matrix of size $n$, then $UU^T = I_n$.

(d) True or False : The sum of two eigenvectors of a matrix $A$ with the same eigenvalue $\lambda$ is also an eigenvector of $A$.

(e) True or False : If $A$ is a $7 \times 9$ matrix with a two-dimensional null space, then the rank of $A$ is 5.

Have a good break!