MATH 314 May 5, 2002

FINAL EXAM

Name :
SSN(last four digits) :
Section: 11:30-12:20, MWF
Instructor : Jon-Lark Kim

Cheating will be punished. Show your work logically and write neatly. Just guessing answer won't get any credit. **Write your work within the space given below, which will be possible!**

Problem 1. Define a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$T(x_1, x_2) = (x_1 - 2x_2,\ x_1 + 3x_2,\ -3x_1 + 2x_2).$$

(a) (10pt) Find the standard matrix A for T.

(b) (10pt) Find $\mathbf{x} = (x_1, x_2)$ such that $T(\mathbf{x}) = (8, -7, -12)$.

Problem 2. Let

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -3 & 4 & 1 \\ 2 & 4 & -3 \end{bmatrix}.$$

(a) (15pt) Find the inverse of A.

(b) (5pt) Double check your answer by multiplying A by its inverse to get an identity matrix.

Problem 3. Assume that matrix A and B are row equivalent.

$$A = \begin{bmatrix} 1 & 4 & -3 & -1 & 9 \\ -2 & -6 & 6 & -1 & -10 \\ -3 & -6 & 9 & -6 & -3 \\ 3 & 4 & -9 & 9 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -3 & 5 & -7 \\ 0 & 2 & 0 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (15pt) Find bases for $\mathrm{Col}(A)$, $\mathrm{Row}(A)$, and $\mathrm{Nul}(A)$.

(b) (5pt) Find also $\mathrm{rank}A$ and dimension of $\mathrm{Nul}(A)$ and then show it satisfies the rank theorem.

Problem 4. Let $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. Answer the following.

(a) (5pt) Find all eigenvalues of A by finding the characteristic equation of A. Include the multiplicity of each eigenvalue.

(b) (10pt) Find a basis for each corresponding eigenspace of A.

(c) (5pt) Determine whether A is diagonalizable.

Problem 5. Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$.

(a) (5pt) Determine whether A and B are diagonalizable. Justify your answer to get the full credit.

(b) (15pt) If A or B is diagonalizable, then find P and D so that PDP^{-1} is A or B.

Problem 6. Let $\{\mathbf{x}_1, \mathbf{x}_2\}$ be a basis for W =span$\{\mathbf{x}_1, \mathbf{x}_2\}$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix},$$

(a) (10pt) Find an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ for W.

(b) (10pt) Let $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Write $\hat{\mathbf{y}} = \text{Proj}_W \mathbf{y}$ in term of your new orthogonal basis obtained in (a).

Problem 7. (a) (15pt) Find an orthogonal basis for the column space of A, where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

(b) (5pt) Double-check whether your basis is really orthogonal.

Problem 8. Let

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(a) (5pt) Show that the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent *by inspection.*

(b) (15pt) Find all least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$.

(c) (10pt) Compute the least-squares error associated with the least-squares solutions found in (a).

Problem 9. (a) (5pt) Give a definition of the similarity of two n by n matrices A, B.

(b) (5pt) Then prove that if two n by n matrices A, B are similar, they have the same characteristic polynomial.

Problem 10. (20pt) Circle True or False for each statement. If true give a brief justification and if false give a counterexample, to get full credit. Each is worth 4 points.

(a) True or False : Every vector space with dimension at least 2 has an orthogonal basis.

(b) True or False : If the least-squares error of a matrix equation $A\mathbf{x} = \mathbf{b}$ is zero, then $A\mathbf{x} = \mathbf{b}$ is inconsistent.

(c) True or False : If U is an n by n matrix such that $U^T U = I_n$, where I_n is an identity matrix of size n, then $UU^T = I_n$.

(d) True or False : The sum of two eigenvectors of a matrix A with the same eigenvalue λ is also an eigenvector of A.

(e) True or False : If A is a 7×9 matrix with a two-dimensional null space, then the rank of A is 5.

Have a good break!