Cheating will be punished. Show your work logically and write neatly. Just guessing answer won’t get any credit. Write your work within the space given below, which will be possible!

Problem 1. The set of the first four Laguerre polynomials is given
\[ \mathcal{B} = \{1, 1 - t, 2 - 4t + t^2, 6 - 18t + 9t^2 - t^3\}. \]

Note that \( \mathcal{B} \) forms a basis for \( \mathbb{P}_3 \).

(a) (10pt) Find the coordinate vector of \( 2 - 4t + t^2 \) relative to \( \mathcal{B} \).

(b) (10pt) Find the coordinate vector of \( p(t) = 6 - 15t + 8t^2 - t^3 \) relative to \( \mathcal{B} \).
**Problem 2.** Assume that matrix $A$ and $B$ are row equivalent.

\[
A = \begin{bmatrix}
1 & 1 & -3 & 7 & 9 & -9 \\
1 & 2 & -4 & 10 & 13 & -12 \\
1 & -1 & -1 & 1 & 1 & -3 \\
1 & -3 & 1 & -5 & -7 & 3 \\
1 & -2 & 0 & 0 & -5 & -4 \\
\end{bmatrix},
B = \begin{bmatrix}
1 & 1 & -3 & 7 & 9 & -9 \\
0 & 1 & -1 & 3 & 4 & -3 \\
0 & 0 & 0 & 1 & -1 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(a) (15pt) Find bases for $\text{Col}(A)$, $\text{Row}(A)$, and $\text{Nul}(A)$.

(b) (5pt) Find also $\text{rank}(A)$ and dimension of $\text{Nul}(A)$. Then check whether your answer satisfies the rank theorem.
Problem 3. (5 pt) By using the definition of a subspace, prove that for an \( m \times n \) matrix \( A \), \( \text{Nul}(A) \) is a subspace of \( \mathbb{R}^n \).

Problem 4. Let

\[ H = \{ (a, b, c, d) : a - 2b - c = 0, \ b - 3d = 0, \ 2c + d = 0 \} \].

(a) (2pt) Find a matrix \( A \) so that \( H \) is the nullspace of \( A \).

(b) (3pt) Find a basis for \( H \) and the dimension of \( H \).
Problem 5. Let $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ be bases of the vector space $\mathbb{R}^2$. Answer the following.

(a) (10pt) Let $b_1 = 3c_1 + 4c_2$ and $b_2 = -c_1 + 2c_2$. Find $[x]_\mathcal{C}$ for $x = 4b_1 - 2b_2$ by computing $P_{\mathcal{C} \leftarrow \mathcal{B}}$.

(b) (10pt) Let

$$b_1 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad 2_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

Find $P_{\mathcal{C} \leftarrow \mathcal{B}}$ and $P_{\mathcal{B} \leftarrow \mathcal{C}}$. 

Problem 6. Let

\[ A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}. \]

(a) (5pt) Find all eigenvalues of \( A \).

(b) (15pt) Find a basis for each corresponding eigenspace of \( A \).
**Problem 7 (10 pt).** Circle True or False for each statement. Then in either case, justify your answer to get full credit. Each is worth 2 points.

(a) True or False: Since $A\mathbf{0} = \mathbf{0}$ for a square matrix $A$, $\mathbf{0}$ is an eigenvector of $A$.

(b) True or False: If $c\mathbf{u} = \mathbf{0}$ for some scalar $c \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{R}^n$, then $\mathbf{u}$ must be $\mathbf{0}$.

(c) True or False: If for a given $m \times n$ matrix $A$, the equation $A\mathbf{x} = \mathbf{b}$ is consistent for any $\mathbf{b} \in \mathbb{R}^m$, then $\text{rank} A = n$.

(d) True or False: A single nonzero vector by itself is linearly dependent.

(e) True or False: The dimension of $\text{Row}(A)$ can be different from the dimension of $\text{Col}(A)$.
From Old Mid I (you can see these from my webpage)

Problem 3. Let

\[ A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \]

(a) (10pt) Find the determinant of \( A \) by cofactor expansion and row operation methods. Check whether you got the same result.

True or False Question

(d) True or False : \( \det(AB) = \det(A^T B^T) \).

(e) True or False : If two rows of an \( n \times n \) matrix \( A \) are identical, then \( \det(A) = 0 \).