RELATION BETWEEN WEIGHT DISTRIBUTION AND COMBINATORIAL IDENTITIES

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Abstract. The purpose of this paper is to give a new method to derive many combinatorial identities by using the weight distributions of binary linear codes. This method is described as follows. We choose a binary linear code and its dual code both of which have simplified weight distributions. Then we substitute these weight distributions into the MacWilliams equations and the Pless power moments and obtain interesting combinatorial identities as well as simple proofs of some well-known identities. We apply this method to certain binary linear codes.

§1. INTRODUCTION AND PASCAL'S TRIANGLE

The theory of error-correcting codes has become a fascinating mathematical topic [6] since Shannon published the papers [8] in 1948. MacWilliams and Pless found the MacWilliams equations and the Pless power moments respectively in 1963 as the names imply[3,5]. These identities relate the weight distribution of a code $\mathcal{C}$ to the weight distribution of the dual code $\mathcal{C}^\perp$ having many applications in coding theory. These identities imply that we can find the the unknown weight distribution of $\mathcal{C}^\perp$ when the weight distribution of $\mathcal{C}$ is known. For example, we can find the exact weight distribution of the binary Hamming code $[2^r-1, 2^r-1-r, 3]$ since its dual code is the binary simplex code $[2^r-1, r, 2^{r-1}]$ where any nonzero vector has constant weight 2^{r-1}[4].

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However our application of these identities is a converse of what is described above. That is, we start with a code C and its dual $\mathcal{C}^{\perp}$ with *known* weight distribution, and then we substitute these weight distributions into the MacWilliams equations and the Pless power moments deriving combinatorial identities.

For coding theory terms see [4,7]. All codes are binary and linear. An $[n,k,d]$(or $[n,k]$ for short) *code* $\mathcal{C}$ has length n, dimension k, and (minimum) distance exactly d, and is a subspace of $\mathbb{F}_2^n$, where $\mathbb{F}_2$ is the Galois field with two elements 0 and 1. Any vector in $\mathcal{C}$ is called a *codeword* of $\mathcal{C}$. Let $\mathbf{u}=(u_1\cdots u_n), u_i \in \mathbb{F}_2$ and $\mathbf{v}=(v_1\cdots v_n), v_i\in\mathbb{F}_2$. The *inner product* is defined by $\mathbf{u}\cdot\mathbf{v}=\sum_{i=1}^n u_iv_i$ with the sum evaluated in $\mathbb{F}_2$. The *dual* code $\mathcal{C}^{\perp}=\{\mathbf{v}=(v_1\cdots v_n):\mathbf{u}\cdot\mathbf{v}=0 \text{ for all } \mathbf{u}\in\mathcal{C}\}$ is an $[n,n-k]$ code.

A *generator matrix* for an $[n,k]$ code $\mathcal{C}$ is a k by n matrix G whose rows form a basis for $\mathcal{C}$. A *parity check matrix* H for an $[n,k]$ code $\mathcal{C}$ is a generator matrix for its dual code $\mathcal{C}^{\perp}$. Two codes $\mathcal{C},\mathcal{C}'$ are *equivalent* if there exists a coordinate permutation in S_n(the symmetric group on n elements) sending $\mathcal{C}$ into $\mathcal{C}'$.

The *direct sum* of codes $\mathcal{C}[n,k,d]$ and $\mathcal{C}'[n',k',d']$ is the $[n+n',k+k',\min(d,d')]$ code

$$\mathcal{C}\oplus\mathcal{C}'=\{(u_1\cdots u_nv_1\cdots v_n):(u_1\cdots u_n)\in\mathcal{C},(v_1\cdots v_n)\in\mathcal{C}'\}.$$

The (Hamming) *weight* of a codeword is the number of nonzero coordinates. The *weight distribution* of $\mathcal{C}$ consists of the numbers $A_i(i=0,\cdots,n)$, where A_i counts the number of codewords of weight i in $\mathcal{C}$. The *weight enumerator* of $\mathcal{C}$ is the polynomial

$$w(\mathcal{C})=w(\mathcal{C};x)=\sum_{i=0}^{n}A_ix^i.$$

The spaces $\mathbb{F}_2^n, \mathbf{1}=[1\cdots 1]$ and their duals are called *trivial* codes.

Let $\mathcal{C}$ be a binary linear $[n,k]$ code. From now on, A_i denotes the number of vectors of weight i in $\mathcal{C}$ and B_i the number of vectors of weight i in $\mathcal{C}^{\perp}$. Now we introduce the MacWilliams equations [7, p.126].

$$(1)\qquad \sum_{j=0}^{n}\binom{j}{\nu}A_j=2^{k-\nu}\sum_{j=0}^{n}(-1)^j\binom{n-j}{n-\nu}B_j,\quad \nu=0,1,\cdots,n.$$

Also we introduce the Pless power moments [7, p.130].

$$(2)\quad \sum_{j=0}^{n}j^rA_j=\sum_{j=0}^{n}(-1)^jB_j\left[\sum_{\nu=0}^{r}\nu!S(r,\nu)2^{k-\nu}\binom{n-j}{n-\nu}\right],\quad r=0,1,\cdots,$$

where $S(r,\nu)$ is the Stirling number of the second kind. $S(r,\nu)$ satisfies the following;

$$\begin{aligned} &S(0,0)=1, S(r,0)=0 \text{ for } r\neq 0, \text{ and } S(1,1)=1,\\ &S(r+1,\nu)=\nu S(r,\nu)+S(r,\nu-1) \text{ for } r\geq \nu\geq 1. \end{aligned}$$

Pascal's triangle is shown as follows;

$$\begin{array}{ccccccccccccc} &&&&&1&&1&&&&& \\ &&&&1&&2&&1&&&& \\ &&&1&&3&&3&&1&&& \\ &&1&&4&&6&&4&&1&& \\ &1&&5&&10&&10&&5&&1& \\ 1&&6&&15&&20&&15&&6&&1 \end{array} \qquad \begin{array}{ccccccccccccc} &&&&&\binom{1}{0}&&\binom{1}{1}&&&&& \\ &&&&\binom{2}{0}&&\binom{2}{1}&&\binom{2}{2}&&&& \\ &&&\binom{3}{0}&&\binom{3}{1}&&\binom{3}{2}&&\binom{3}{3}&&& \\ &&\binom{4}{0}&&\binom{4}{1}&&\binom{4}{2}&&\binom{4}{3}&&\binom{4}{4}&& \\ &\binom{5}{0}&&\binom{5}{1}&&\binom{5}{2}&&\binom{5}{3}&&\binom{5}{4}&&\binom{5}{5}& \\ \binom{6}{0}&&\binom{6}{1}&&\binom{6}{2}&&\binom{6}{3}&&\binom{6}{4}&&\binom{6}{5}&&\binom{6}{6} \end{array}$$

We note that any nth row of the right Pascal's triangle has numbers of the form $\binom{n}{i}$. Let $\mathcal{C}=\mathbb{F}_2^n$ with generator matrix $G=I_n$. Then it is easy to see that

$$\binom{n}{i}=A_i.$$

This is because the number of vectors of weight i is exactly the number of ways we choose i rows of the generator matrix

$$I_n=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & & \vdots\\ 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Thus we can understand Pascal's triangle in terms of the weight distribution of the binary code consisting of all the vectors of $\mathbb{F}_2^n$ $(n=1,2,3,\cdots)$. This observation makes it possible to use the MacWilliams equations and the Pless power moments.

Our method is described as follows. We choose a binary linear code and its dual code both of which have simplified weight distributions. Then we substitute these weight distributions into the MacWilliams equations and the Pless power moments and obtain interesting combinatorial identities as well as simple proofs of some well-known identities. In section 2 we consider two codes $\mathbb{F}_2^n$ and the even weight code which is the dual of the repetition code $[1 \cdots 1]$. We derive identities, some of which are known, directly by using the MacWilliams equations and the Pless power moments. In section 3 we consider two other codes. One is a code $\mathcal{C}_l$ which has exactly *one* codeword of each weight and the other is the repetition of this code, which is a subcode of the even weight code with exactly *one* codeword of each even weight. It turns out that these codes and their duals have simplified weight distributions by coding theoretical techniques, hence we can apply our method, giving interesting identities. In section 4 we give a short conclusion.

§ 2. RESULTS FROM TRIVIAL CODES

First let us take $\mathcal{C}$ as the whole space $\mathbb{F}_2^n$. So $\mathcal{C}$ is an $[n,n]$ code. Clearly $\mathcal{C}^\perp = \{(0,0,\cdots,0)\}$, which is an $[n,0]$ code. Thus $B_0 = 1$ but $B_j = 0$ for $j = 1,2,\cdots,n$. We have

$$A_j = \binom{n}{j}, j = 0,1,\cdots,n.$$

Then we get the following.

Proposition 1.

$$\sum_{j=0}^{n} \binom{j}{\nu}\binom{n}{j} = 2^{n-\nu}\binom{n}{n-\nu}, \quad \nu = 0,1,\cdots,n,$$

$$\text{and } \binom{0}{\nu} = 2^{-\nu}\sum_{j=0}^{n}(-1)^j\binom{n-j}{n-\nu}\binom{n}{j}, \quad \nu = 0,1,\cdots,n.$$

Proof. Note that the dimension of $\mathcal{C}$ is n. Now we substitute

$$A_j = \binom{n}{j}, j = 0,1,\cdots,n$$

into the left side of equation (1) and $B_0 = 1, B_i = 0 (i = 1,\ldots,n)$ into the right side of equation (1). Then the first identity follows. If we switch A_j and B_j, then the second identity follows. $\square$

Remark 1. The first identity in Proposition 1 can also be proved by using

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k} \text{(see [1, p.150])}.$$

Proposition 2.

$$\sum_{j=0}^{n} j^r \binom{n}{j} = \sum_{\nu=0}^{r} \nu! S(r,\nu) 2^{n-\nu} \binom{n}{n-\nu}, \quad r = 0, 1, \cdots,$$

$$\text{and} \quad 0 = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left[\sum_{\nu=0}^{r} \nu! S(r,\nu) 2^{-\nu} \binom{n-j}{n-\nu} \right], \quad r = 0, 1, \cdots.$$

Proof. This proof is the same as the proof of Proposition 1 except that we use equation (2). □

Remark 2. We remark that the first identity in Proposition 2 can be derived by referring to [2, p.106], which expresses j^r as

$$\sum_{\nu=0}^{r} \nu! S(r,\nu) \binom{j}{\nu}.$$

Next we consider the binary even $[n, n-1]$ code which consists of all even weight codewords of length n. Its dual is the repetition code $[1 \dots 1]$ which is an $[n, 1]$ code. We see that $B_0 = B_n = 1$ and $B_j = 0$ $(j \neq 0, n)$, and that

$$A_{2j} = \binom{n}{2j} \text{ and } A_j = 0 \text{ if } j \text{ is odd.}$$

By using the MacWilliams equations (1) first with A_j and B_j as above and then switching A_j and B_j, we get Proposition 3.

Proposition 3.

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{2j}{\nu} \binom{n}{2j} = 2^{n-1-\nu} \left[\binom{n}{\nu} + (-1)^n \binom{0}{n-\nu} \right], \nu = 0, \cdots, n,$$

$$\text{and} \quad \binom{0}{\nu} + \binom{n}{\nu} = 2^{1-\nu} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-2j}{n-\nu} \binom{n}{2j}, \quad \nu = 0, \cdots, n.$$

Similarly by using the Pless power moments (2) first with A_j and B_j as above and then switching A_j and B_j, we get Proposition 4.

Proposition 4. *For* $r = 0, 1, \cdots,$

$$\sum_{j=0}^{\lfloor n/2 \rfloor} (2j)^r \binom{n}{2j} = \sum_{\nu=0}^{r} \nu! S(r,\nu) 2^{n-1-\nu} \left[\binom{n}{\nu} + (-1)^n \binom{0}{n-\nu} \right]$$

$$\text{and} \quad n^r = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} \left[\sum_{\nu=0}^{r} \nu! S(r,\nu) 2^{1-\nu} \binom{n-2j}{n-\nu} \right].$$

§3. RESULTS FROM NONTRIVIAL CODES

3.1 A binary linear code $\mathcal{C}_l$

Now we construct a binary linear code $\mathcal{C}_l$ of length $2^l - 1$ which has exactly *one* codeword of each weight. We define the following generator matrices inductively.

$$
\begin{aligned}
&G_1 = [1],\\
&G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},\\
&G_3 = G_2 \oplus [\,1 \quad 1 \quad 1 \quad 1\,],\\
&\vdots\\
&G_l = G_{l-1} \oplus [\underbrace{1 \ldots 1}_{2^{l-1}}].
\end{aligned}
$$

Then the corresponding parity check matrices are

$$
\begin{aligned}
&H_1 = [0],\\
&H_2 = [\,0 \quad 1 \quad 1\,],\\
&H_3 = H_2 \oplus [\,1 \quad 1 \quad 1 \quad 1\,]^{\perp},\\
&\vdots\\
&H_l = H_{l-1} \oplus [\underbrace{1 \ldots 1}_{2^{l-1}}]^{\perp}.
\end{aligned}
$$

It is easy to see that the code $\mathcal{C}_l$ with generator matrix G_l has the weight distribution $A_0 = A_1 = \cdots = A_{2^l-1} = 1$ and $\mathcal{C}_l$ is a linear binary $[2^l - 1, l, 1]$ code. Also we note that the dual $\mathcal{C}_l^{\perp}$ has generator matrix $H_l = H_{l-1} \oplus [1 \ldots 1]^{\perp}$ and hence the weight enumerator of $\mathcal{C}_l^{\perp}$ is equal to

$$
\begin{aligned}
(3)\ & w(\mathcal{C}_{l-1}^{\perp}; x) \left[\sum_{k=0}^{2^{(l-2)}} \binom{2^{l-1}}{2k} x^{2k} \right] \\
&= \left[\binom{2}{0} + \binom{2}{2} x^2 \right] \left[\binom{4}{0} + \binom{4}{2} x^2 + \binom{4}{4} x^4 \right] \cdots \left[\sum_{k=0}^{2^{(l-2)}} \binom{2^{l-1}}{2k} x^{2k} \right].
\end{aligned}
$$

We can find the weight distribution $B_0, B_1, \ldots, B_{2^l-1}$ of $\mathcal{C}_l^{\perp}$, as follows.

Lemma 5. *Let $n = 2^l - 1$ and let j be $0 \le j \le 2^l - 1$. Then we get*

$$B_j = \sum_{\substack{m_1+\cdots+m_{l-1}=j \\ m_i \ge 0,\text{even}}}^{n-1} \binom{2}{m_1}\binom{2^2}{m_2}\cdots\binom{2^{l-1}}{m_{l-1}} \ \textit{if } j \textit{ is even}, \tag{4}$$

and $B_j = 0$ *if j is odd.*

Proof. Suppose j is even. We note that B_j is the coefficient of x^j in the expansion of the equation (3). Also we see that the coefficient of x^{2i} in each weight enumerator in the right side of (3) has the form $\binom{2^k}{2i}$. Therefore if we expand the right side of (3) and combine like terms, then the coefficient of x^j is exactly the right side of (4), as required. Suppose j is odd. Then since the weight enumerator of H_2 is $1+x^2$, $[1\ldots1]^\perp$ consists of all even weight codewords of an appropriate length and then the recursive definition of H_l is used, we have that the values of B_j(j odd) are zero. This completes the proof. $\square$

From Lemma 5 and the Pless power moments (2) we get the following result.

Proposition 6. *Let $n = 2^l - 1$ and B_j of the form (4). Then we get*

$$\sum_{j=0}^{n} j^r = \sum_{j=0,\text{even}}^{n} B_j \left[\sum_{\nu=0}^{r} \nu! S(r,\nu) 2^{l-\nu} \binom{n-j}{n-\nu}\right], \ r = 0, 1, \cdots,$$

$$\sum_{j=0,\text{even}}^{n} j^r B_j = \sum_{j=0}^{n} (-1)^j \left[\sum_{\nu=0}^{r} \nu! S(r,\nu) 2^{n-l-\nu} \binom{n-j}{n-\nu}\right], \ r = 0, 1, \cdots.$$

Proof. For the first identity we consider the code $\mathcal{C}_l$ defined above. Then we use equation (2) with $A_0 = \cdots = A_{2^l-1} = 1, B_j$ in (4) and the dimension of $\mathcal{C}_l$ equal to l. Then the first identity follows. The second identity follows by switching A_j and B_j. $\square$

Let us take an example to see the first identity in Proposition 6.

Example. Let $n = 7$. So $l = 3$. Then the weight distribution of $\mathcal{C}_3^\perp$ is $B_0 = 1, B_2 = B_4 = 7$ and $B_6 = 1$ since by Lemma 5,

$$B_0 = \binom{2}{0}\binom{4}{0} = 1,$$

$$B_2 = \binom{2}{0}\binom{4}{2} + \binom{2}{2}\binom{4}{0} = 7,$$

$$B_4 = \binom{2}{0}\binom{4}{4} + \binom{2}{2}\binom{4}{2} = 7,$$

$$\text{and } B_6 = \binom{2}{2}\binom{4}{4} = 1.$$

Therefore for any nonnegative integer r with the convention $0^0 = 1$, we have

$$\sum_{j=0}^{7} j^r = \left[\sum_{\nu=0}^{r} \nu! S(r,\nu) 2^{3-\nu} \binom{7}{7-\nu}\right] + 7\left[\sum_{\nu=0}^{r} \nu! S(r,\nu) 2^{3-\nu} \binom{7-2}{7-\nu}\right] + 7\left[\sum_{\nu=0}^{r} \nu! S(r,\nu) 2^{3-\nu} \binom{7-4}{7-\nu}\right] + \left[\sum_{\nu=0}^{r} \nu! S(r,\nu) 2^{3-\nu} \binom{7-6}{7-\nu}\right].$$

Similarly we can derive another identity by considering $\mathcal{C}_l$ and the MacWilliams equations (1).

Proposition 7. *Let $n = 2^l - 1$ and B_j of the form (4). Then we get*

$$\binom{n+1}{\nu+1} = 2^{l-\nu} \sum_{j=0,\text{even}}^{n} \binom{n-j}{n-\nu} B_j, \quad \nu = 0, 1, \cdots, n,$$

and $$\sum_{j=0,\text{even}}^{n} \binom{j}{\nu} B_j = 2^{n-l-\nu} \sum_{j=0}^{n} (-1)^j \binom{n-j}{n-\nu}, \quad \nu = 0, 1, \cdots, n.$$

Proof. For the first identity we consider $\mathcal{C}_l$ defined above. We also use the fact that B_i(i odd) are zero, which was shown in Lemma 5. Let's look at the MacWilliams equations (1). The left hand side of (1) is simplified as follows,

$$\text{LHS of (1)} = \sum_{j=0}^{n} \binom{j}{\nu} = \binom{n+1}{\nu+1}$$

since all A_i's are 1. Also since B_i's(i odd) are zero, we get

$$\text{RHS of (1)} = 2^{l-\nu} \sum_{j=0,\text{even}}^{n} \binom{n-j}{n-\nu} B_j, \quad \nu = 0, 1, \cdots, n.$$

For the second identity we switch A_j and B_j. $\square$

The first identity in Proposition 6 gives a partial result for finding a general formula for $\sum_{j=0}^{n} j^r$ (n is any nonzero integer). It is interesting to compare this identity to the following well-known identity which can be derived by using the expression of j^r in Remark 2.

Fact. *For any nonzero integer n,*

$$\sum_{j=0}^{n} j^r = \sum_{\nu=0}^{r} \nu! S(r,\nu) \binom{n+1}{\nu+1}, \quad r = 0, 1, \cdots . \tag{5}$$

Remark 3. If we combine the first identities in Proposition 6 and Proposition 7(respectively), then we can get Fact (5) with the assumption that $n = 2^l - 1$.

3.2 A binary linear code $\mathcal{C}_{l,l}$

So far we have considered the code $\mathcal{C}_l$ with generator matrix G_l defined in the section 3.1. We now think about the repetition of this code $\mathcal{C}_l$, call it $\mathcal{C}_{l,l}$. Then $\mathcal{C}_{l,l}$ has generator matrix $G_{l,l} = [G_l \ G_l]$. We note that $\mathcal{C}_l$ is an $[n = 2^l - 1, l, 1]$ code and that $H_l = H_{l-1} \oplus [\underbrace{1 \dots 1}_{2^{l-1}}]^\perp$. Thus the parity check matrix for $G_{l,l}$ is

$$H_{l,l} = \begin{bmatrix} I_n & I_n \\ H_l & 0 \end{bmatrix},$$

$$\text{which is equivalent to } \left[\begin{array}{cc|cc} I_{2^{l-1}-1} & I_{2^{l-1}-1} & \multicolumn{2}{c}{0} \\ H_{l-1} & 0 & & \\ \hline & & I_{2^{l-1}} & I_{2^{l-1}} \\ & 0 & [1\dots1]^\perp & 0 \end{array}\right]$$

$$= H_{l-1,l-1} \oplus \text{the even } [2^l, 2^l - 1] \text{ code}.$$

So we can find the weight distribution B'_j of $\mathcal{C}_{l,l}^\perp$ as well as the weight distribution A'_j of $\mathcal{C}_{l,l}$.

Lemma 8. *Let $m = (2^l - 1)2 = 2^{l+1} - 2$ be the length of the code $\mathcal{C}_{l,l}$. Let j be $0 \le j \le 2^{l+1} - 2$. Then*

$$\begin{aligned} B'_j &= \sum_{\substack{m_1+\cdots+m_l=j \\ m_i \ge 0,\text{even}}}^{m} \binom{2}{m_1}\binom{2^2}{m_2}\cdots\binom{2^l}{m_l} \ (j \text{ even}), \\ B'_j &= 0 \ (j \text{ odd}), \\ \text{and} \quad A'_j &= 1 \ (j \text{ even}), A'_j = 0 \ (j \text{ odd}). \end{aligned}$$

Proof. The proof about B'_j is similar to the proof of Lemma 5. The proof about A'_i is clear by the construction of $\mathcal{C}_{l,l}$. $\square$

Then we can get identities similar to Proposition 6.

Proposition 9. *Let $m = (2^l - 1)2$ and B'_j of the form in Lemma 8. Then we get*

$$\sum_{j=0,\mathrm{even}}^{m} j^r = \sum_{j=0,\mathrm{even}}^{m} B'_j \left[\sum_{\nu=0}^{r} \nu! S(r,\nu) 2^{l-\nu} \binom{m-j}{m-\nu}\right], \ r = 0, 1, \cdots,$$
$$\sum_{j=0,\mathrm{even}}^{m} j^r B'_j = \sum_{j=0,\mathrm{even}}^{m} \left[\sum_{\nu=0}^{r} \nu! S(r,\nu) 2^{m-l-\nu} \binom{m-j}{m-\nu}\right], \ r = 0, 1, \cdots.$$

We also get identities similar to Proposition 7.

Proposition 10. *Let $m = (2^l - 1)2$ and B'_j of the form in Lemma 8. Then we get*

$$\sum_{j=0,\mathrm{even}}^{m} \binom{j}{\nu} = 2^{l-\nu} \sum_{j=0,\mathrm{even}}^{m} \binom{m-j}{m-\nu} B'_j, \quad \nu = 0, 1, \cdots, m,$$
$$\text{and} \sum_{j=0,\mathrm{even}}^{m} \binom{j}{\nu} B'_j = 2^{m-l-\nu} \sum_{j=0,\mathrm{even}}^{m} \binom{m-j}{m-\nu}, \quad \nu = 0, 1, \cdots, m.$$

§4. CONCLUSION

We showed how to get interesting combinatorial identities by using the MacWilliams equations and the Pless power moments. Using other codes and their duals whose weight distributions are simplified, we can get further identities. This shows that the MacWilliams equations and the Pless power moments in coding theory are very useful in deriving combinatorial identities.

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