Formally self-dual additive codes over $\mathbb{F}_4$ and near-extremal self-dual codes

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Abstract

We introduce a class of formally self-dual additive codes over $\mathbb{F}_4$ as a natural analogue of binary formally self-dual codes, which is missing in the study of additive codes over $\mathbb{F}_4$. We define extremal formally self-dual additive codes over $\mathbb{F}_4$ and classify all such codes. Interestingly, we find exactly three formally self-dual additive $(7,2^7)$ odd codes over $\mathbb{F}_4$ with minimum distance $d = 4$, a better minimum distance than any self-dual additive $(7,2^7)$ codes over $\mathbb{F}_4$. We further define near-extremal formally self-dual additive codes over $\mathbb{F}_4$ as an analogue of near-extremal binary formally self-dual codes, and prove that they do not exist if their lengths are $n = 16, 18$ or $n \geq 20$. We improve the bounds on the minimum distance of formally self-dual binary codes in a similar manner. Finally, we extend S. Zhang’s best known upper bound on the highest minimum distance of the four types of classical self-dual codes of large lengths.

Key Words: Additive codes; extremal codes; near-extremal codes; formally self-dual additive codes; self-dual codes.  
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Abbreviated title: Near-extremal formally self-dual codes
1 Introduction

Binary self-dual codes and additive self-dual codes over $\mathbb{F}_4$ have common properties such as Type I, Type II, shadow codes, $s$-extremal codes, etc [9],[19]. Binary formally self-dual codes are defined as a class of binary codes whose weight enumerators are the same as the weight enumerators of their dual codes. Hence they include the class of binary self-dual codes, and their weight enumerators are combinations of Gleason polynomials of Type I [13].

One of the motivations studying binary formally self-dual codes is that some binary formally self-dual codes (e.g., at lengths 10 or 18 [13]) have a better minimum distance than any self-dual codes of the same length. This observation leads us to consider a class of formally self-dual additive codes over $\mathbb{F}_4$ and to find their highest minimum distances using their extremal or near-extremal weight enumerators.

The class of formally self-dual additive codes can be put together with the four types of classical self-dual codes (i.e., Type I binary self-dual codes, Type II binary self-dual codes, Type III ternary self-dual codes, and Type IV Hermitian self-dual codes) since the weight enumerators of these five classes are generated by two Gleason polynomials.

In this paper, we introduce a class of formally self-dual additive codes over $\mathbb{F}_4$ and classify them. We find an upper bound on the highest minimum distance of these codes. In an analogously manner, we improve a best known upper bound on the highest minimum distance of formally self-dual binary codes. We extend a best known upper bound on the highest minimum distance of the four types of classical self-dual codes of large lengths.

This paper is organized as follows. Section 2 gives a brief introduction to additive codes over $\mathbb{F}_4$ and defines extremal formally self-dual additive even or odd codes over $\mathbb{F}_4$.

Section 3 classifies extremal formally self-dual additive odd codes of length up to 7 and shows that there is no extremal formally self-dual additive odd code of length $n \geq 8$. In particular, we construct exactly three formally self-dual additive $(7,2^7)$ odd codes over $\mathbb{F}_4$ with minimum distance $d = 4$, a better minimum distance than any self-dual additive $(7,2^7)$ codes over $\mathbb{F}_4$. These $(7,2^7,4)$ additive codes over $\mathbb{F}_4$ would produce binary $[28,14,7]$ codes or optimal binary $[28,14,8]$ codes via Construction O or Construction E respectively, as described in [14].

In Section 4, we describe possible weight enumerators of formally self-dual additive odd codes with even length. Our results are $\mathbb{F}_4$-analogues of near-extremal formally self-dual binary codes considered in [15]. We show that there exist near-extremal formally self-dual additive codes of length 6 with all possible weight enumerators.

Section 5 shows that given an $(n,2^n,d)$ formally self-dual additive code, if $n = 18$ or $n \geq 20$, then $d < \left[ \frac{n}{2} \right]$, i.e., there is no near-extremal formally self-dual additive code. We do this by showing that $A_{\left[ \frac{n}{2} \right]+2} < 0$. It is further shown in Section 7 that there is no near-extremal formally self-dual additive code of length 16.

Similarly, in Section 6 we improve the bounds on the minimum distance of binary formally self-dual codes. More precisely, we show that there is no near-extremal formally self-dual $[n = 8t + 2l, \frac{n}{2}, d = 2\left[ \frac{n}{8} \right]]$ binary linear code if $l = 1$ and $t \geq 12$, if $l = 2$ and $t \geq 13$, or if $l = 3$ and $t \geq 14$, which, together with the case $l = 0$ and $t \geq 9$ considered in [8], gives the best known upper bounds on the length $n$ for which there exist near-extremal formally self-dual binary codes.
Section 7 deals with the nonexistence of near-extremal binary f.s.d. even codes, Type II self-dual codes, Type III self-dual codes, Type IV self-dual codes, and f.s.d.a. odd codes over $\mathbb{F}_4$. Our main result (Theorem 7.2) extends the S. Zhang’ best known bounds [20] on the highest minimum distance of four types of classical self-dual codes of large lengths (see also [16, Sec. 11.1]).

2 Preliminaries

We refer to [2], [6], [12] for definitions and facts about additive codes over $\mathbb{F}_4$.

An additive code $C$ of length $n$ over $\mathbb{F}_4$ is an additive subgroup of $\mathbb{F}_4^n$. $C$ contains $2^k$ codewords for some $0 \leq k \leq 2n$, and can be defined by a $k \times n$ generator matrix, with entries from $\mathbb{F}_4$, whose rows span $C$ additively. We call $C$ an $(n, 2^k)$ code. We denote $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$, where $\omega^2 = \omega + 1$. The conjugation of $x \in \mathbb{F}_4$ is defined by $x^* = x^2$. The trace map, $Tr : \mathbb{F}_4 \rightarrow \mathbb{F}_2$, is defined by $Tr(x) = x + x^*$. The Hermitian trace inner product of two vectors $u = (u_1, u_2, \cdots, u_n)$ and $v = (v_1, v_2, \cdots, v_n)$ in $\mathbb{F}_4^n$ is given by

$$<u, v> = Tr(u \cdot v) = \sum_{i=1}^{n} Tr(u_i v_i^*) = \sum_{i=1}^{n} (u_i v_i^2 + u_i^2 v_i) \pmod{2}.$$ 

We define the dual of the code $C$ with respect to the Hermitian trace inner product by $C^\perp = \{ u \in \mathbb{F}_4^n | <u, c> = 0 \text{ for all } c \in C \}$. Then $C^\perp$ is also additive. $C$ is self-orthogonal if $C \subseteq C^\perp$. If $C = C^\perp$, then $C$ is self-dual and must be an $(n, 2^n)$ code. Two additive codes $C_1$ and $C_2$ are equivalent if there is a map sending the codewords of $C_1$ onto the codewords of $C_2$ where the map consists of a permutation of coordinates followed by a possible scaling of coordinates by nonzero elements of $\mathbb{F}_4$ followed by possible conjugation of some of the coordinates. The automorphism group $\text{Aut}(C)$ of $C$ is the group of all maps sending $C$ to itself using these three operations.

The Hamming weight of $u$, denoted $wt(u)$, is the number of nonzero components of $u$. The Hamming distance between $u$ and $v$ is $wt(u - v)$. The minimum distance of the code $C$ is the minimal Hamming distance between any two distinct codewords of $C$. Since $C$ is an additive code, the minimum distance is also given by the smallest nonzero weight of any codeword in $C$. A code with minimum distance $d$ is called an $(n, 2^k, d)$ code. The weight distribution of the code $C$ is the sequence $(A_0, A_1, \ldots, A_n)$, where $A_i$ is the number of codewords of weight $i$. The weight enumerator of $C$ is the polynomial

$$W_C(x, y) = \sum_{i=0}^{n} A_i x^{n-i} y^i.$$ 

Theorem 2.1. (MacWilliams’ identity) Let $C$ be an additive code over $\mathbb{F}_4$. Then

$$W_{C^\perp}(x, y) = \frac{1}{|C|} W_C(x + 3y, x - y).$$

Proof. See Theorem 1 in [10].
Definition 2.2. An additive code $C$ over $F_4$ is formally self-dual (f.s.d.) if

$$W_C(x, y) = W_C(x, y).$$

A formally self-dual additive (f.s.d.a.) code $C$ over $F_4$ is even if all the weights of codewords of $C$ is divisible by 2, and odd if some of the weights of codewords of $C$ is not divisible by 2.

Theorem 2.3. Let $C$ be an f.s.d.a. $(n, 2^n)$ code over $F_4$. Then the weight enumerator $W_C(x, y)$ is a weighted homogeneous polynomial of weight $n$ in $x + y$ and $y(x - y)$.

Proof. The proof is essentially the same as the one in Theorem 3 of [10].

Theorem 2.4. An f.s.d.a. even code $C$ is self-dual.

Proof. Let $u, v \in C$. By the following identity

$$\text{wt}(u + v) - \text{wt}(u) - \text{wt}(v) \equiv <u, v> \pmod{2},$$

the theorem holds.

Remark 2.5. By Theorem 2.4, we focus on f.s.d.a. odd codes over $F_4$.

Let $C$ be an $(n, 2^n, d)$ f.s.d.a. code over $F_4$. Define $m = \lfloor n/2 \rfloor$. By Theorem 2.3 the weight enumerator of a code $C$ can be written as

$$W_C(x, y) = \sum_{i=0}^{m} a_i (x + y)^{n-2i} (y(x - y))^i$$

(1)

with unique integral numbers $a_i$. There is a unique choice of the numbers $a_0, \ldots, a_m$ such that the right hand side of (1) equals

$$x^n + 0 \cdot x^{n-1}y + \cdots + 0 \cdot x^{n-m}y^m + A_{m+1}x^{n-m-1}y^{m+1} + \cdots + A_n y^n.$$  

(2)

We call (2) the extremal weight enumerator and a code with this extremal weight enumerator an extremal code. So, an extremal code has minimal weight $d \geq \lfloor n/2 \rfloor + 1$.

Theorem 2.6. The minimal distance $d$ of an f.s.d.a. code $C$ over $F_4$ of length $n$ satisfies

$$d \leq \lfloor n/2 \rfloor + 1.$$  

Proof. The proof is essentially the same as the one in Theorem 11 of [10].

3 Classification of extremal formally self-dual additive odd codes over $F_4$

In this section we classify extremal $(n, 2^n, \lfloor n/2 \rfloor + 1)$ f.s.d.a. odd codes. We consider the following construction method, which is a modified balance principal for self-dual codes over $F_4$ [6] and for formally self-dual binary codes [5].
Lemma 3.1. Let $\mathcal{C}$ be an extremal f.s.d.a. $(n, 2^n)$ odd code with minimum distance $d = \lceil \frac{n}{2} \rceil + 1$ and $G$ be its $n$ by $n$ generator matrix. Assume that $n$ is odd. Then $G$ is equivalent to the following matrix.

$$G' = \begin{bmatrix} I_d & B \\ D & E \end{bmatrix}$$

where $I_d$ is the $d$ by $d$ identity matrix, $D$ is the $(d-1)$ by $d$ matrix of the form

$$D = \begin{bmatrix} \omega & \omega & 0 & 0 & \cdots & 0 \\ \omega & 0 & \omega & 0 & \cdots & 0 \\ \vdots & & & & & \\ \omega & 0 & 0 & \cdots & 0 & \omega \end{bmatrix},$$

$B$ is a $2$ by $(d-1)$ matrix of one of the following forms

$$B_1 = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega \end{bmatrix} \text{ or } B_2 = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \omega & \omega & \cdots & \omega \end{bmatrix},$$

and $E$ is an $\mathbb{F}_4$-matrix of size $(n-2) \times (d-1)$.

Proof. We may assume that $\mathcal{C}^\perp$ contains the all-one vector $1$ of minimum distance $d$ up to equivalence. Then the dual of the code generated by $1$ is the code $F$ generated by the rows of $I_d$ and $D$. Note that $F$ has $\mathbb{F}_2$-dimension $n = 2d - 1$. If any proper nonzero subcode of $F$ is used in the left side of $G'$ then there is a vector $y = (0|x)$ in $\mathcal{C}$ with $x \neq 0$ since the $\mathbb{F}_2$-dimension of $\mathcal{C}$ is $n$. As $1 \leq \text{wt}(y) \leq d - 1$, we get a contradiction. Therefore the left side of $G'$ must be generated by $I_d$ and $D$. Finally, $B_1$ or $B_2$ is chosen up to equivalence to keep the minimum distance of the code generated by the first two rows of $G'$ to be at least $d$.

The most time consuming part of Lemma 3.1 is to fill in the entries of $E$. We do this by Magma [3] using the equivalence of additive codes developed in [6]. We have the following result.

- $n = 1$ : $W_C(1, y) = 1 + y$ : There is a unique f.s.d.a. $(1, 2, 1)$ code with generator matrix $(1)$. This is self-dual.

- $n = 2$ : $W_C(1, y) = 1 + 3y^2$ : There is no extremal f.s.d.a. odd code of length 2. Only one extremal f.s.d.a. even code generated by $(11)$ and $(\omega \omega)$ exists [10]. This is a Type II self-dual code. It is easy to check by hand that there are, up to equivalence, exactly two f.s.d.a. $(2, 2^2, 1)$ non self-dual codes, generated by $\{(1 0), (\omega 0)\}$ or $\{(1 0), (\omega 1)\}$, respectively.

- $n = 3$ : $W_C(1, y) = 1 + 3y^2 + 4y^3$ : We show that there are exactly two extremal f.s.d.a. codes of length 3, denoted by $C_{2,1}$ and $C_{2,2}$. They have following generator matrices respectively using Lemma 3.1.
We note that $C_{2,1}$ is a Type I self-dual code \[10\], while $C_{2,2}$ is not. We check that $|\text{Aut}(C_{2,1})| = 8$ and $|\text{Aut}(C_{2,2})| = 24$.

- $n = 4$ : $W_C(1, y) = 1 + 12y^3 + 3y^4$ : There is no $(4, 2^4, 3)$ additive self-dual code \[10\]. Modifying the left side of $G'$ given in Lemma 3.1, we easily obtain a unique extremal f.s.d.a. code $C_4$ of length 4, which has $|\text{Aut}(C_4)| = 36$, and whose generator matrix is unique up to equivalence as shown below.

\[
G(C_4) = \begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
\bar{w} & \bar{w} & 0 & \bar{w} \\
\bar{w} & \bar{w} & 0 & \bar{w} \\
\end{bmatrix}.
\]

Therefore $n = 4$ is the first length for which a f.s.d.a. code has a better minimum distance than any self-dual additive code over $\mathbb{F}_4$ of that length.

The code $C_4$ is also a linear code over $\mathbb{F}_4$ generated by $(1, 1, 0, 1)$ and $(1, 0, 1, \omega)$. Note that $C_4$ regarded as a $\mathbb{F}_4$-linear code is not Euclidean self-dual. We can further choose a Euclidean self-dual f.s.d.a. odd code over $\mathbb{F}_4$ as follows. Let $C$ be a linear code over $\mathbb{F}_4$ by the following generator matrix.

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & \omega & \bar{w} \\
\end{bmatrix}.
\]

Then by Section 3.2 in \[19\], $C$ is a Euclidean self-dual code with the weight enumerator

\[
W_C(1, y) = 1 + 12y^3 + 3y^4.
\]

Since the MacWilliams identity of Euclidean self-dual code is the same as f.s.d.a. codes over $\mathbb{F}_4$, $C$ is a $(4, 2^4, 3)$ f.s.d.a. codes over $\mathbb{F}_4$. It is straightforward to check that $C$ is equivalent to $C_4$ as an additive code.

- $n = 5$ : $W_C(1, y) = 1 + 10y^3 + 15y^4 + 6y^5$ : There are exactly four $(5, 2^5, 3)$ f.s.d.a. (non self-dual) codes, denoted by $C_{5,1}, \ldots, C_{5,4}$ and a unique $(5, 2^5, 3)$ Type I self-dual code $C_{5,5}$. Their generator matrices $G(C_{5,i})$ for $i = 1, \ldots, 5$ are given below, and their automorphism group orders are all 16. $C_{5,5}$ must be equivalent to the unique $(5, 2^5, 3)$ Type I self-dual code \[10\].

\[
G(C_{5,1}) = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & w \\
0 & 0 & 1 & w & 1 \\
w & w & 0 & 1 & w \\
w & 0 & w & 1 & w^2 \\
\end{bmatrix}, \quad G(C_{5,2}) = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & w \\
0 & 0 & 1 & w & 1 \\
w & w & 0 & 1 & w \\
w & 0 & w & w & w^2 \\
\end{bmatrix},
\]
\[
G(C_{5,3}) = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & w \\
0 & 0 & 1 & w & 1 \\
w & w & 0 & w^2 & 1 \\
w & 0 & w & 1 & w^2
\end{bmatrix}, \quad G(C_{5,4}) = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & w \\
0 & 0 & 1 & w & 1 \\
w & w & 0 & w & 1 \\
w & 0 & w & 1 & w^2
\end{bmatrix}.
\]

\[
G(C_{5,5}) = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & w & w \\
0 & 0 & 1 & w^2 & w^2 \\
w & w & 0 & w & 1 \\
w & 0 & w & 1 & w^2
\end{bmatrix}.
\]

- \( n = 6 \): \( W_C(1, y) = 1 + 45y^4 + 18y^6 \): Due to this weight enumerator, there is no extremal f.s.d.a. odd code over \( \mathbb{F}_4 \). It is known that there is a unique Type II self-dual code of length 6 [10]. Modifying the construction of Lemma 3.1, we construct 235 f.s.d.a. \((6, 2^6)\) odd non self-dual codes with \( d = 3 \) and the unique Type I self-dual \((6, 2^6, 3)\) code [10] up to equivalence, making use of restricted equivalence described in [6].

- \( n = 7 \): \( W_C(1, y) = 1 + 35y^4 + 42y^5 + 28y^6 + 22y^7 \): Each case of Lemma 3.1 produces exactly three inequivalent \((7, 2^7, 4)\) f.s.d.a. (non self-dual) codes. As the three codes of the first case of Lemma 3.1 using \( B_1 \), denoted by \( C_{7,1}, \ldots, C_{7,3} \) are equivalent to those of the second case using \( B_2 \), we only display their generator matrices below, and their automorphism group orders are 7, 6, 42, respectively. There is no \((7, 2^7, 4)\) Type I self-dual code but there exist four \((7, 2^7, 3)\) Type I self-dual codes (note: the three codes with these parameters in Table 1 of [6] are corrected in [4]). Thus just like \( n = 4 \) case, the minimum distance of extremal f.s.d.a. codes of length \( n = 7 \) beats that of any self-dual codes of the same length. Hence applying Construction O or Construction E [14] to the three extremal f.s.d.a. codes we get binary \([28, 14, 7]\) codes or optimal binary \([28, 14, 8]\) codes [1].
• $n = 8$ : negative weight enumerator $(A_{\frac{n}{2}+2} < 0)$. Hence there is no extremal f.s.d.a. code of length 8.

• $n = 9$ : $W_C(1, y) = 1 + 126y^5 + 84y^6 + 108y^7 + 171y^8 + 22y^9$ : Using Lemma 3.1, we have checked that there is no extremal f.s.d.a. code of length 9.

• $n = 10$ : negative weight enumerator $(A_{\frac{n}{2}+2} < 0)$. Hence there is no extremal f.s.d.a. code of length 10.

• $n = 11$ : $W_C(1, y) = 1 + 462y^6 + 495y^7 + 880y^8 + 66y^9 + 144y^{10}$ : Using Lemma 3.1, we have checked that there is no extremal f.s.d.a. code of length 11.

• $n \geq 12$ : $A_{\frac{n}{2}+2} < 0$ by the proof of Theorem 12 in [10]. Hence there is no extremal f.s.d.a. code of length $n$ if $n \geq 12$.

In particular, we have shown the following.

**Theorem 3.2.**

(i) There exists a unique extremal f.s.d.a. odd $(4, 2^4, 3)$ code over $F_4$.

(ii) There exist exactly three extremal f.s.d.a. odd $(7, 2^7, 4)$ codes over $F_4$.

(iii) Any f.s.d.a. odd $(n, 2^n, d)$ code over $F_4$ satisfies

$$d \leq \left\lfloor \frac{n}{2} \right\rfloor$$

for $n \geq 8$.

Hence it is natural to consider the following definition.

**Definition 3.3.** An f.s.d.a. odd code over $F_4$ of length $n$ with minimum distance $d = \left\lfloor \frac{n}{2} \right\rfloor$ is called near-extremal.

The above results are summarized in Table 1. Here the second column $d_{\text{non sd fsd ao}}^{\text{non sd fsd ao}}$ refers to the (extremal (E) or near-extremal (NE)) minimum distance of a possible formally self-dual additive odd codes excluding Type I self-dual codes, the third column refers to the number of the corresponding codes, and the forth and fifth column refer to the minimum distance of optimal Type I codes and the number of the corresponding codes respectively from [4],[6],[10],[11].

### 4 Possible weight enumerators of near-extremal f.s.d.a. odd codes over $F_4$ with even length

In this section we calculate the possible weight enumerators of f.s.d.a. odd codes with even length. Our results are $F_4$-analogues of near-extremal formally self-dual binary codes done in [15]. The approach in this section is similar to that of [15].

Let $C$ be an f.s.d.a. odd code over $F_4$. We define a code in $C$ as even code if its weight is even and odd code if its weight is odd. We denote the set of even codes in $C$ by $EC$ and
Table 1: Highest minimum distance of formally self-dual additive odd (f.s.d.a.o.) non self-dual codes over $\mathbb{F}_4$ of lengths up to 12

<table>
<thead>
<tr>
<th>length</th>
<th>$d_{fsdao}^{non sd}$</th>
<th>num$^{non sd}_{fsdao}$</th>
<th>$d_{sd,I}$</th>
<th>num$^{sd,I}_{([4],[6],[10],[11])}$</th>
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</thead>
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<tr>
<td>2</td>
<td>1NE</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2E</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>3E</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
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<td>3E</td>
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<td>1</td>
</tr>
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<td>6</td>
<td>3NE</td>
<td>235</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>4E</td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>4NE</td>
<td>$\geq 10$ [7]</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>$\leq 4NE$</td>
<td>?</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>10</td>
<td>$5NE$</td>
<td>$\geq 4$ [7]</td>
<td>4</td>
<td>101</td>
</tr>
<tr>
<td>11</td>
<td>$\leq 5NE$</td>
<td>?</td>
<td>5</td>
<td>1</td>
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<td>$6NE$</td>
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</tr>
<tr>
<td>13</td>
<td>$\leq 6NE$</td>
<td>?</td>
<td>5</td>
<td>$\geq 9$</td>
</tr>
<tr>
<td>14</td>
<td>6 or 7NE</td>
<td>594 [7] or ?</td>
<td>5 or 6</td>
<td>$\geq 5$ or ?</td>
</tr>
</tbody>
</table>

the set of odd codes in $\mathcal{C}$ by $\text{OC}$. We call an f.s.d.a. odd code balanced if it contains the same number of even codes and odd codes. By Theorem 2.3 we have

$$W_{\mathcal{C}}(x, y) = \sum_{k=0}^{[n/2]} a_i (x + y)^{n-2k} (y(x - y))^k$$ (3)

for some $a_i$. And

$$W_{\mathcal{C}}(1, -1) = |\text{EC}| - |\text{OC}|.$$ (4)

From (3) and (4), we have

**Proposition 4.1.** If $\mathcal{C}$ is a f.s.d.a. odd code over $\mathbb{F}_4$ with odd length is balanced.

Let $\mathcal{C}$ be a near-extremal f.s.d.a. odd code over $\mathbb{F}_4$ with even length $n$. Then the coefficients $a_0, a_1, \ldots, a_{n/2-1}$ in (3) are uniquely determined. We denote the coefficient $a_{\frac{n}{2}}$ in (3) as $\alpha$. Then

$$W_{\mathcal{C}}(1, -1) = \alpha (-2)^{\frac{n}{2}} = (-1)^{\frac{n}{2}} \cdot \alpha \cdot 2^{\frac{n}{2}}.$$ 

So we have

$$|\text{EC}| - |\text{OC}|$$ if and only if $\alpha = 0.$ (5)

From (5), we have

**Proposition 4.2.** The weight distribution of a near-extremal f.s.d.a. odd code over $\mathbb{F}_4$ with even length and $|\text{EC}| = |\text{OC}|$ is unique, and is given by (3) with $\alpha (= a_{\frac{n}{2}}) = 0.$

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Now we want to calculate the possible values of $\alpha$. Before we do that, we need the following results which are stated in [17].

A binary linear code is called even if it only contains even weight vectors. A doubly-even (d.e.) vector has weight $\equiv 0 \pmod{4}$, while a singly-even (s.e.) vector has weight $\equiv 2 \pmod{4}$. A hyperbolic plane is a two dimensional space generated by two doubly-even vectors which are not orthogonal to each other. An anisotropic plane is generated by two singly-even non-orthogonal vectors. We write $C_1 \perp C_2$ to mean the vector space direct sum of two codes $C_1$ and $C_2$ which are orthogonal to each other. If $C$ is a code, let $R(C)$ denote the largest doubly-even subcode of $C \cap C^\perp$ and let $r = \dim R(C)$. Let $a$ denote the number of d.e. vectors in $C$ and $b$ denote the number of s.e. vectors in $C$. Then every even binary linear $[n, k]$ code $C$ is one of three types.

(i) Hyperbolic Type. Here $C = R(C) \perp H_{2m}$ where $H_{2m}$ is the orthogonal sum of $m$ hyperbolic planes. Clearly $k = r + 2m$. In this case the following holds

\[
a = 2^r (2^{2m-1} + 2^{m-1}), \\
b = 2^r (2^{2m-1} - 2^{m-1}).
\]

(ii) Anisotropic Type. Here $C = R(C) \perp H_{2(m-1)} \perp A$ where $H_{2(m-1)}$ is the orthogonal sum of $(m-1)$ hyperbolic planes and $A$ is an anisotropic plane. Again $k = r + 2m$. Further,

\[
a = 2^r (2^{2m-1} - 2^{m-1}), \\
b = 2^r (2^{2m-1} + 2^{m-1}).
\]

(iii) Odd Anisotropic Type. Here $C = R(C) \perp H_{2m} \perp < x >$ where $x$ is a singly-even vector. Now $k = r + 2m + 1$ and $a = b = 2^{k-1}$.

Now we are ready to prove the following theorem.

**Theorem 4.3.** Let $C$ be an $(n, 2^n, \left[\frac{n}{2}\right])$ near-extremal f.s.d.a. odd code over $\mathbb{F}_4$ with even length. Then the possible coefficient $\alpha (= a_\frac{n}{2})$ in (3) is given by

\[\alpha = 0 \quad \text{or} \quad \pm 2^i \quad \text{for} \quad i = 0, 1, 2, \ldots, \frac{n}{2} - 1.\]

Furthermore, $|EC|$ and $|OC|$ are given by

\[|EC| = 2^{n-1} + \alpha 2^{\frac{n}{2} - 1}, \]
\[|OC| = 2^{n-1} - \alpha 2^{\frac{n}{2} - 1}.\]

**Proof.** If $\alpha = 0$, then the theorem holds. So, we assume that $\alpha \neq 0$. Define $\phi : \mathbb{F}_4 \to \mathbb{F}_2^3$ by

\[\phi(0) = (0, 0, 0), \]
\[\phi(1) = (1, 1, 0), \]
\[\phi(\omega) = (1, 0, 1), \]
\[\phi(\omega) = (0, 1, 1).\]
Define $\phi_n : \mathbb{F}_4^n \to \mathbb{F}_2^{3n}$ by

$$\phi_n(a_1, a_2, \ldots, a_n) = (\phi(a_1), \phi(a_2), \ldots, \phi(a_n)).$$

Then $\phi_n$ is $\mathbb{F}_2$-linear, and $\phi_n(\mathcal{C})$ is a $[3n, n, 2[\frac{n}{2}]$ binary linear even code. Let $a$ be the number of doubly-even codes in $\phi_n(\mathcal{C})$ and $b$ be the number of singly-even codes in $\phi_n(\mathcal{C})$. Then we have $|\mathcal{EC}| = a$ and $|\mathcal{OC}| = b$. Using the notations before Theorem 4.3, we have

$$n = r + 2m,$$

$$a - b = \pm 2^{r + m} = (-1)^{\frac{r}{2}} a 2^{\frac{n}{2}},$$

$$a + b = 2^n.$$

So, $r$ is even and

$$a = 2^{n-1} + (-1)^{\frac{n}{2}} a 2^{\frac{n}{2} - 1} \text{ and } b = 2^{n-1} - (-1)^{\frac{n}{2}} a 2^{\frac{n}{2} - 1}.$$  

And

$$\alpha = \pm (-1)^{\frac{n}{2}} 2^{\frac{n}{2}}, \left(\frac{r}{2} = 0, 1, \ldots, \frac{n}{2}\right).$$

Now we only have to prove that $\frac{r}{2} \neq \frac{n}{2}$. Suppose $\frac{r}{2} = \frac{n}{2}$. If $\phi_n(\mathcal{C})$ is Hyperbolic Type, then $b = 0$. But this is impossible since $\mathcal{C}$ is f.s.d.a. odd code over $\mathbb{F}_4$. If $\phi_n(\mathcal{C})$ is Anisotropic Type, then $\phi_n(\mathcal{C}) = R(\phi_n(\mathcal{C}))$. This is also impossible. So, $\frac{r}{2} \neq \frac{n}{2}$. \hfill $\square$

Now we state some possible weight enumerators with $\alpha = a_2$ for small code length.

- $n = 6$ :
  $W(1, y) = 1 + (8 + \alpha)y^3 + (21 - 3\alpha)y^4 + (24 + 3\alpha)y^5 + (10 - \alpha)y^6$. The 235 f.s.d.a. odd codes over $\mathbb{F}_4$ of length 6 with $d = 3$ in Section 3 produce all possible values of $\alpha = -4, -2, -1, 0, 1, 2, 4$. We give only seven codes with each $\alpha$ from $-4$ to $4$, denoted by $\mathcal{C}_{6,1}, \mathcal{C}_{6,2}, \ldots, \mathcal{C}_{6,7}$, respectively. Their generator matrices are given in Table 2.

- $n = 8$ :
  $W(1, y) = 1 + (26 + \alpha)y^4 + (64 - 4\alpha)y^5 + (72 + 6\alpha)y^6 + (64 - 4\alpha)y^7 + (29 + \alpha)y^8$. It is shown [7] that there exist f.s.d.a. odd codes over $\mathbb{F}_4$ with $\alpha = -8, -2, 1, 4$.

- $n = 10$ :
  $W(1, y) = 1 + (92 + \alpha)y^5 + (170 - 5\alpha)y^6 + (200 + 10\alpha)y^7 + (295 - 10\alpha)y^8 + (220 + 5\alpha)y^9 + (46 - \alpha)y^{10}$. It is shown [7] that there exist f.s.d.a. odd codes over $\mathbb{F}_4$ with $\alpha = -2, 1, 4$.

- $n = 12$ :
  $W(1, y) = 1 + (332 + \alpha)y^6 + (384 - 6\alpha)y^7 + (525 + 15\alpha)y^8 + (1280 - 20\alpha)y^9 + (1020 + 15\alpha)y^{10} + (384 - 6\alpha)y^{11} + (170 + \alpha)y^{12}$. It is shown [7] that there exist f.s.d.a. odd codes over $\mathbb{F}_4$ with $\alpha = -2$.

- $n = 14$ :
  $W(1, y) = 1 + (1220 + \alpha)y^7 + (469 - 7\alpha)y^8 + (1596 + 21\alpha)y^9 + (5348 - 35\alpha)y^{10} + (3388 + 35\alpha)y^{11} + (2226 - 21\alpha)y^{12} + (1988 + 7\alpha)y^{13} + (148 - \alpha)y^{14}$. No near-extremal f.s.d.a. odd code over $\mathbb{F}_4$ of this length is known.

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Table 2: Near-extremal formally self-dual additive odd codes over $\mathbb{F}_4$ of length 6 with $\alpha = -4, -2, -1, 0, 1, 2, 4$, respectively

$$G(C_{6,1}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & w & w \\ 0 & 1 & 0 & w & w & 0 \\ 0 & 0 & 1 & w^2 & w & w^2 \\ w & w & 0 & 0 & 1 & 0 \\ w & 0 & w & 1 & 0 & 0 \end{bmatrix}, \quad G(C_{6,2}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & w & w \\ 0 & 1 & 0 & w & w & 0 \\ 0 & 0 & 1 & w^2 & w^2 & w^2 \\ w & w & 0 & 0 & 1 & 0 \\ w & 0 & w & 1 & 0 & 0 \end{bmatrix},$$

$$G(C_{6,3}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & w & w \\ 0 & 1 & 0 & w & w & 0 \\ 0 & 0 & 1 & w & w & 0 \\ w & w & 0 & 1 & 1 & 0 \\ w & 0 & w & w & 0 & 0 \end{bmatrix}, \quad G(C_{6,4}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & w & w \\ 0 & 1 & 0 & w & w & 0 \\ 0 & 0 & 1 & w & w & 0 \\ w & w & 0 & 0 & 1 & 0 \\ w & 0 & w & 1 & 0 & 0 \end{bmatrix},$$

$$G(C_{6,5}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & w & w \\ 0 & 1 & 0 & w & w & 0 \\ 0 & 0 & 1 & w^2 & w^2 & w^2 \\ w & w & 0 & 1 & 0 & 0 \\ w & 0 & w & 0 & 1 & 0 \end{bmatrix}, \quad G(C_{6,6}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & w & w \\ 0 & 1 & 0 & w & w & 0 \\ 0 & 0 & 1 & w & w & 0 \\ w & w & 0 & w^2 & 0 & 0 \\ w & 0 & w & w^2 & 0 & 0 \end{bmatrix},$$

$$G(C_{6,7}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & w & w \\ 0 & 1 & 0 & w & 1 & 0 \\ 0 & 0 & 1 & w & 0 & 0 \\ w & w & 0 & w & 2 & 0 \\ w & 0 & w & w & 0 & 0 \end{bmatrix}. $$
5 Nonexistence of near-extremal f.s.d.a. odd codes over $F_4$

We study whether there exists a near-extremal f.s.d.a. odd code over $F_4$ of length $n \geq 15$. In fact, we have the following.

**Theorem 5.1.** Let $C$ be an $(n, 2^n, d)$ f.s.d.a. odd code over $F_4$. If $n = 18$ or $n \geq 20$, then $d < \lfloor \frac{n}{2} \rfloor$. In other words, there is no near-extremal f.s.d.a. odd code over $F_4$ if $n = 18$ or $n \geq 20$.

**Proof.** We prove this theorem by showing that $A_{\lfloor \frac{n}{2} \rfloor + 2} < 0$ if $n = 18$ or $n \geq 20$. Define $m = \lfloor \frac{n}{2} \rfloor$. Suppose $C$ be a near-extremal f.s.d.a. odd code with $d = m$. From (1) we have the following weight enumerator of $C$.

$$
\sum_{i=0}^{m} a_i (1 + y)^{n-2i}(y(1-y))^i = 1 + A_m y^m + \cdots + A_n y^n.
$$

Using the B"{u}rman-Lagrange formula [18], we have the following equation.

$$
\frac{1}{(1+y)^n} = \sum_{i=0}^{m} a_i \left( \frac{y(1-y)}{(1+y)^2} \right)^i - \frac{1}{(1+y)^n} (A_m y^m + \cdots + A_n y^n)
= \sum_{i=0}^{\infty} \alpha_i \left( \frac{y(1-y)}{(1+y)^2} \right)^i,
$$

where $\alpha_0 = 1$ and for $i \geq 1$

$$
\alpha_i = \frac{1}{i} \left[ \text{coeff. of } y^{i-1} \text{ in } \left\{ \left( \frac{1}{(1+y)^n} \right) \left( \frac{y(1+y)^2}{1-y} \right)^i \right\} \right].
$$

(6)

We have

$$
a_i = \alpha_i, (i = 0, 1, 2, \ldots, m - 1)
$$

and

$$
\sum_{i=m}^{\infty} \alpha_i \left( \frac{y(1-y)}{(1+y)^2} \right)^i = a_m \left( \frac{y(1-y)}{(1+y)^2} \right)^m = \frac{1}{(1+y)^n} (A_m y^m + \cdots + A_n y^n).
$$

(7)

The left hand side of (7) is equal to

$$
\alpha_m y^m + (\alpha_{m+1} - 3mA_m) y^{m+1} + \left( \alpha_{m+2} - (3m+3)\alpha_{m+1} + \left( \frac{2m+1}{2} \right) + 2m^2 + \left( \frac{m}{2} \right) \alpha_m \right) y^{m+2} + O(y^{m+3}).
$$

(8)
The right hand side of (7) is equal to
\[
(a_m - A_m) y^m + (-3ma_m + nA_m - A_{m+1}) y^{m+1} \\
+ \left( \left( \binom{2m+1}{2} + 2m^2 + \binom{m}{2} \right) a_m - \left( \frac{n+1}{2} \right) A_m + nA_{m+1} - A_{m+2} \right) y^{m+2} \\
+ O(y^{m+3}).
\] (9)

From (8) and (9), we have
\[
A_m = a_m - \alpha_m, \\
A_{m+1} = -\alpha_{m+1} + (n - 3m) A_m,
\] (10)
\[
A_{m+2} = A_m \left\{ - \left( \frac{n+1}{2} \right) + n(n-3m) + \left( \frac{2m+1}{2} \right) + 2m^2 + \binom{m}{2} \right\} \\
+ (3m + 3 - n) \alpha_{m+1} - \alpha_{m+2}.
\] (11)

Since \( A_{m+1} \geq 0 \), we have the following from (10).
\[
A_m \leq \frac{\alpha_{m+1}}{n - 3m}.
\] (12)

Now we want to show that \( A_{m+2} < 0 \). We prove this fact by two cases, i.e., \( n = 2m \) and \( n = 2m + 1 \). In the first, we assume that \( n = 2m \). Then from (11), we have the following using (12).
\[
A_{m+2} = A_m \cdot \frac{m(m-1)}{2} + (m+3) \alpha_{m+1} - \alpha_{m+2} \\
\leq -\frac{\alpha_{m+1}}{m} \cdot \frac{m(m-1)}{2} + (m+3) \alpha_{m+1} - \alpha_{m+2} \\
= \frac{m+7}{2} \cdot \alpha_{m+1} - \alpha_{m+2}.
\] (13)

It is enough to show that
\[
\frac{m+7}{2} \cdot \alpha_{m+1} < \alpha_{m+2}.
\]

From (6), we have the following.
\[
\alpha_{m+1} = \frac{-2m}{m+1} \left( \binom{2m-1}{m-1} + \binom{2m}{m} \right),
\]
\[
\alpha_{m+2} = \frac{-2m}{m+2} \left( \binom{2m-1}{m-2} + 3 \binom{2m}{m-1} + 3 \binom{2m+1}{m} + \binom{2m+2}{m+1} \right).
\]

It is sufficient to prove that
\[
\frac{m+7}{2} (m+2) \left( \binom{2m-1}{m-1} + \binom{2m}{m} \right) \\
> (m+1) \left( \binom{2m-1}{m-2} + 3 \binom{2m}{m-1} + 3 \binom{2m+1}{m} + \binom{2m+2}{m+1} \right).
\]
By using the following well-known identity
\[
\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+r}{r} = \binom{n+r+1}{r},
\] (14)
we have
\[
\binom{2m-1}{m-2} + 3\binom{2m}{m-1} + 3\binom{2m+1}{m} < 3\binom{2m+2}{m} < 3\binom{2m+2}{m+1}.
\]
So, it is sufficient to prove that
\[
\frac{m+7}{2}\binom{2m}{m} > 4\binom{2m+2}{m+1}.
\]
It is sufficient to prove that
\[
m\binom{2m}{m} > 8\binom{2m+2}{m+1}.
\] (15)
(15) is equivalent to
\[
m(m-31) > 16.
\] (16)
So, if \( m \geq 32 \), then \( A_{m+2} < 0 \). For \( 9 \leq m \leq 31 \), we can check directly \( A_{m+2} < 0 \) from (13). Now we assume that \( n = 2m+1 \). This case is similar to the case \( n = 2m \).
\[
A_{m+2} = A_m \cdot \frac{m(m-3)}{2} + (m+2)\alpha_{m+1} - \alpha_{m+2}
\]
\[
\leq \frac{\alpha_{m+1} \cdot m(m-3)}{n-3m} + (m+2)\alpha_{m+1} - \alpha_{m+2}
\]
\[
= \alpha_{m+1} \cdot \left( m + 2 + \frac{m(m-3)}{2(-m+1)} \right) - \alpha_{m+2}.
\] (17)
It is enough to show that
\[
\alpha_{m+1} \cdot \left( m + 2 + \frac{m(m-3)}{2(-m+1)} \right) < \alpha_{m+2}.
\]
From (6), we have the following.
\[
\alpha_{m+1} = -\frac{(2m+1)}{m+1} \binom{2m}{m},
\]
\[
\alpha_{m+2} = -\frac{(2m+1)}{m+2} \left( \binom{2m}{m-1} + 2\binom{2m+1}{m} + \binom{2m+2}{m+1} \right).
\]
It is sufficient to prove that
\[
\binom{2m}{m} \left( m + 2 + \frac{m(m-3)}{2(-m+1)} \right)
\]
\[
> \left( \binom{2m}{m-1} + 2\binom{2m+1}{m} + \binom{2m+2}{m+1} \right).
\]
Note that
\[ m + 2 + \frac{m(m - 3)}{2(-m + 1)} > \frac{m}{2}. \]
So, it is sufficient to prove that
\[ \left( \frac{2m}{m} \right) \frac{m}{2} > 3 \left( \frac{2m + 2}{m + 1} \right). \]
So, if \( m \geq 24 \), then \( A_{m+2} < 0 \). For \( 10 \leq m \leq 23 \), we can check directly \( A_{m+2} < 0 \) from (17).
Now we have proved the theorem. \( \Box \)

6 Nonexistence of near-extremal formally self-dual binary linear codes

In this section we prove nonexistence of near-extremal \([n, \frac{n}{2}, 2[\frac{n}{8}]]\) formally self-dual (f.s.d.) binary linear codes using a similar method to Section 5.

**Theorem 6.1.** Let \( C \) be an \([n, \frac{n}{2}, d]\) f.s.d. binary linear code. Let \( n = 8t + 2l \) (\( l = 1, 2, 3 \)).

(i) If \( l = 1 \) and \( t \geq 12 \), then \( d < 2[\frac{n}{8}] \).

(ii) If \( l = 2 \) and \( t \geq 13 \), then \( d < 2[\frac{n}{8}] \).

(iii) If \( l = 3 \) and \( t \geq 14 \), then \( d < 2[\frac{n}{8}] \).

In other words, there is no near-extremal f.s.d. binary linear code if \( l = 1 \) and \( t \geq 12 \), if \( l = 2 \) and \( t \geq 13 \), or if \( l = 3 \) and \( t \geq 14 \).

**Remark 6.2.** For completeness, we present the following. Let \( C \) be an \([n, \frac{n}{2}, d]\) f.s.d. binary linear code. Let \( n = 8t + 2l \) (\( l = 0, 1, 2, 3 \)). Then combining the above theorem and the results in [8] proving the conjecture of [15] for \( t \geq 9 \), we have that if \( l = 0 \) and \( t \geq 9 \), if \( l = 1 \) and \( t \geq 12 \), if \( l = 2 \) and \( t \geq 13 \), or if \( l = 3 \) and \( t \geq 14 \), then
\[ d < 2 \left[ \frac{n}{8} \right]. \]

**Proof.** (of Theorem 6.1) Suppose \( C \) be a near-extremal f.s.d. binary code with \( d = 2t \). We have the following weight enumerator.
\[ W_C(1, y) = \sum_{k=0}^{t} c_k f^{4t+l-4k} y^k = 1 + \sum_{k=t}^{4t+l} A_{2k} y^k, \]
where \( f = 1 + y, g = y(1 - y)^2 \), and \( c_k \) are integers. Using the Bürman-Lagrange formula [18], we have the following equation.
\[ f^{-4t-l} = \sum_{k=0}^{t} c_k \phi^k - f^{-4t-l} \sum_{k=t}^{4t+l} A_{2k} y^k \]
\[ = \sum_{s=0}^{\infty} \alpha_s \phi^s, \]
\[ 16 \]
where \( \varphi = \frac{g}{f t}, \alpha_0 = 1 \) and for \( i \geq 1 \)

\[
\alpha_s = \frac{1}{s} \left[ \text{coeff. of } y^{s-1} \text{ in } \left( f^{-4t+l} \frac{y}{\varphi} \right)^s \right]. \tag{18}
\]

We have

\[
\alpha_i = c_i, \quad (0 \leq i < t).
\]

and

\[
\sum_{s=t}^{\infty} \alpha_s f^{4t-4s} g^s = c_t g^t - f^{-l} \sum_{k=t}^{4t+l} A_{2k} y^k. \tag{19}
\]

The left hand side of (19) is equal to

\[
\alpha_t y^t + (-2t\alpha_t + \alpha_{t+1}) y^{t+1} + \left( \frac{2t}{2} \alpha_t + (-2t - 6)\alpha_{t+1} + \alpha_{t+2} \right) y^{t+2} + O(y^{t+3}). \tag{20}
\]

The right hand side of (19) is equal to

\[
(c_t - A_{2t}) y^t + (-2tc_t - (-lA_{2t} + A_{2t+2})) y^{t+1} + \left( c_t \left( \frac{2t}{2} \right) - \left( A_{2t} \left( \frac{l+1}{2} \right) - lA_{2t+2} + A_{2t+4} \right) \right) y^{t+2} + O(y^{t+3}). \tag{21}
\]

From (20) and (21), we have

\[
A_{2t} = c_t - \alpha_t,
\]

\[
A_{2t+2} = (-2t + l)A_{2t} - \alpha_{t+1},
\]

\[
A_{2t+4} = A_{2t} \left\{ \left( \frac{2t}{2} \right) - \left( \frac{l+1}{2} \right) + l^2 - 2lt \right\} + (2t + 6 - l)\alpha_{t+1} - \alpha_{t+2}. \tag{23}
\]

Since \( A_{2t+2} \geq 0 \), we have the following from (22).

\[
A_{2t} \leq \frac{\alpha_{t+1}}{-2t + l}. \tag{24}
\]

By (23) and (24), we have the following.

\[
A_{2t+4} \leq \frac{\alpha_{t+1}}{-2t + l} \left\{ \left( \frac{2t}{2} \right) - \left( \frac{l+1}{2} \right) + l^2 - 2lt \right\} + (2t + 6 - l)\alpha_{t+1} - \alpha_{t+2}. \tag{25}
\]

We want to show that \( A_{2t+4} < 0 \). Since \( A_{2t} > 0 \), we have \( \alpha_{t+1} < 0 \) from (24). So in (25), we have

\[
\frac{\alpha_{t+1}}{-2t + l} \left\{ \left( \frac{2t}{2} \right) - \left( \frac{l+1}{2} \right) + l^2 - 2lt \right\} < \frac{\alpha_{t+1}}{-2t} (2t^2 - 2t) = \alpha_{t+1}(-t + 1),
\]

\[
\frac{\alpha_{t+1}}{-2t + l} \left\{ \left( \frac{2t}{2} \right) - \left( \frac{l+1}{2} \right) + l^2 - 2lt \right\} + (2t + 6 - l)\alpha_{t+1} \leq \alpha_{t+1}(-t + 1 + 2t + 6 - l) \leq \alpha_{t+1}(t + 4).
\]

It is enough to show that

\[
\alpha_{t+1}(t + 4) > (-\alpha_{t+2}).
\]

We present \( \alpha_{t+1}, \alpha_{t+2} \) for each \( l = 1, 2, 3 \) by (18) in the following.
\( l = 1 \)

\[
\alpha_{t+1} = \frac{-4t - 1}{t + 1} \left\{ \binom{3t + 1}{t} + 2 \binom{3t}{t-1} + \binom{3t - 1}{t-2} \right\},
\]

\[
\alpha_{t+2} = \frac{-4t - 1}{t + 2} \left\{ \binom{3t + 4}{t + 1} + 6 \binom{3t + 3}{t} + 15 \binom{3t + 2}{t - 1} + 20 \binom{3t + 1}{t - 2} + 15 \binom{3t}{t - 3} + 6 \binom{3t - 1}{t - 4} + \binom{3t - 2}{t - 5} \right\}.
\]

\( l = 2 \)

\[
\alpha_{t+1} = \frac{-4t - 2}{t + 1} \left\{ \binom{3t + 1}{t} + \binom{3t}{t-1} \right\},
\]

\[
\alpha_{t+2} = \frac{-4t - 2}{t + 2} \left\{ \binom{3t + 4}{t + 1} + 5 \binom{3t + 3}{t} + 10 \binom{3t + 2}{t - 1} + 10 \binom{3t + 1}{t - 2} + 5 \binom{3t}{t - 3} + \binom{3t - 1}{t - 4} \right\}.
\]

\( l = 3 \)

\[
\alpha_{t+1} = \frac{-4t - 3}{t + 1} \binom{3t + 1}{t},
\]

\[
\alpha_{t+2} = \frac{-4t - 3}{t + 2} \left\{ \binom{3t + 4}{t + 1} + 4 \binom{3t + 3}{t} + 6 \binom{3t + 2}{t - 1} + 4 \binom{3t + 1}{t - 2} + \binom{3t}{t - 3} \right\}.
\]

For all \( l = 1, 2, 3 \), we have the following

\[
-\alpha_{t+1} \geq \frac{4t + 1}{t + 1} \binom{3t + 1}{t},
\]

\[
-\alpha_{t+2} \geq \frac{4t + 3}{t + 2} \left\{ \binom{3t + 4}{t + 1} + 6 \binom{3t + 3}{t} + 15 \binom{3t + 2}{t - 1} + 20 \binom{3t + 1}{t - 2} + 15 \binom{3t}{t - 3} + 6 \binom{3t - 1}{t - 4} + \binom{3t - 2}{t - 5} \right\}.
\]

Note the following simple inequalities. In the first, by (14) we have

\[
20 \binom{3t + 1}{t - 2} + 15 \binom{3t}{t - 3} + 6 \binom{3t - 1}{t - 4} + \binom{3t - 2}{t - 5} \leq 20 \binom{3t + 2}{t - 2}.
\]
And
\[
\binom{3t + 2}{t - 2} \leq \frac{1}{2} \binom{3t + 2}{t - 1},
\]
\[
\binom{3t + 2}{t - 1} \leq \frac{1}{3} \binom{3t + 3}{t},
\]
\[
\binom{3t + 3}{t} \leq \frac{1}{3} \binom{3t + 4}{t + 1}.
\]
Thus
\[
\binom{3t + 4}{t + 1} + 6 \binom{3t + 3}{t} + 15 \binom{3t + 2}{t - 1} + 20 \binom{3t + 1}{t - 2}
\]
\[
+ 15 \binom{3t}{t - 3} + 6 \binom{3t - 1}{t - 4} + \binom{3t - 2}{t - 5} \leq 6 \binom{3t + 4}{t + 1}.
\]
It is enough to show that
\[
(t + 4)^{\frac{4t - 1}{t + 1}} \binom{3t + 1}{t} > \frac{4t + 3}{t + 2} \cdot 6 \cdot \frac{3t + 4}{t + 1}.
\]
It is enough to show that
\[
(t + 4)(4t + 1) \frac{(3t + 1)!}{(2t + 1)!t!} > 6 \cdot (4t + 3) \cdot \frac{(3t + 4)!}{(2t + 3)!(t + 1)!}.
\]
It is enough to show that
\[
(t + 4)(4t + 1)(2t + 3)(2t + 2) > 18 \cdot (4t + 3)(3t + 4)(3t + 2).
\]
It is enough to show that
\[
16t^4 > 18 \cdot 5t \cdot 4t \cdot 4t.
\]
If \( t > 90 \), then (26) holds. For the remaining finite case, we can check \( A_{2t+4} < 0 \) by direct calculation in (25). \( \square \)

7 Nonexistence of near-extremal (formally) self-dual codes

The methods of Sections 5 and 6 can be applied to Type II, Type III, and Type IV self-dual codes so that we can obtain similar results. But in this section, we adopt S. Zhang’s approach in [20] to investigate these self-dual codes. So, the organization of this section is similar to the one in [20]. In [20], the existence problem of extremal self-dual codes is treated by a systematic way with unified notations. In this section, we combine our idea of Sections 5 and 6 with Zhang’s systematic approach and unified notations to solve the nonexistence problem of near-extremal (formally) self-dual codes.
We are interested in the six types of (formally) self-dual codes, i.e., binary f.s.d. even codes, Type II self-dual codes, Type III self-dual codes, Type IV self-dual codes, additive self-dual even codes over $\mathbb{F}_4$, and f.s.d.a. odd codes over $\mathbb{F}_4$. The weight enumerators $W(X,Y)$ of these (formally) self-dual codes are generated by the following two Gleason polynomials respectively [19].

(i) binary f.s.d. even code : $f = X^2 + Y^2, g = X^2Y^2(X^2 - Y^2)^2$,
(ii) Type II self-dual code : $f = X^8 + 14X^4Y^4, g = X^4Y^4(X^4 - Y^4)^4$,
(iii) Type III self-dual code : $f = X^4 + 8XY^3, g = Y^3(X^3 - Y^3)^3$,
(iv) Type IV self-dual code : $f = X^2 + 3Y^2, g = Y^2(X^2 - Y^2)^2$,
(iv)′ additive self-dual even code over $\mathbb{F}_4$ : $f = X^2 + 3Y^2, g = Y^2(X^2 - Y^2)^2$,
(v) f.s.d.a. odd code over $\mathbb{F}_4$ : $f = X + Y, g = Y(X - Y)$.

**Definition 7.1.** A (formally) self-dual code of length $n$ with minimum distance $d$ below is called near-extremal.

(i) binary f.s.d. even code : $d = 2[\frac{n}{8}]$,
(ii) Type II self-dual code : $d = 4[\frac{n}{24}]$,
(iii) Type III self-dual code : $d = 3[\frac{n}{12}]$,
(iv) Type IV self-dual code : $d = 2[\frac{n}{6}]$,
(iv)′ additive self-dual even code over $\mathbb{F}_4$ : $d = 2[\frac{n}{6}]$,
(v) f.s.d.a. odd code over $\mathbb{F}_4$ : $d = [\frac{n}{2}]$.

Now we state the main theorem of this section.

**Theorem 7.2.** There is no near-extremal code with length $n$ for

(i) binary f.s.d. even code : if $n = 8i$ ($i \geq 9$), $8i + 2$ ($i \geq 12$), $8i + 4$ ($i \geq 13$), $8i + 6$ ($i \geq 14$),
(ii) Type II self-dual code : if $n = 24i$ ($i \geq 315$), $24i + 8$ ($i \geq 320$), $24i + 16$ ($i \geq 325$),
(iii) Type III self-dual code : if $m = 12i$ ($i \geq 147$), $12i + 4$ ($i \geq 150$), $12i + 8$ ($i \geq 154$),
(iv) Type IV self-dual code : if $m = 6i$ ($i \geq 38$), $6i + 2$ ($i \geq 41$), $6i + 4$ ($i \geq 43$),
(iv)′ additive self-dual even code over $\mathbb{F}_4$ : if $m = 6i$ ($i \geq 38$), $6i + 2$ ($i \geq 41$), $6i + 4$ ($i \geq 43$),
(v) f.s.d.a. odd code over $\mathbb{F}_4$ : if $n = 2i$ ($i \geq 8$), $2i + 1$ ($i \geq 10$).

In the following proof of Theorem 7.2, we omit Type (iv)′ additive codes as they have the same Gleason polynomials as Type IV self-dual codes so that the proofs of (iv) and (iv)′ are the same.
7.1 Proof of Theorem 7.2

To obtain a unified notation for all the five types we replace $X$ by 1 and $Y^w$ by $y$, and give the following definition: $f = 1 + \alpha y + \delta y^2$, $g = y(1 - y)^w$, where

(i) binary f.s.d. even code : $w = 2, R = 4, S = 2, \alpha = 1, \delta = 0,$
(ii) Type II self-dual code : $w = 4, R = 3, S = 8, \alpha = 14, \delta = 1,$
(iii) Type III self-dual code : $w = 3, R = 3, S = 4, \alpha = 8, \delta = 0,$
(iv) Type IV self-dual code : $w = 2, R = 3, S = 2, \alpha = 3, \delta = 0,$
(v) f.s.d.a. odd code over $\mathbb{F}_4$ : $w = 1, R = 2, S = 1, \alpha = 1, \delta = 0.$

With the unified notation, Gleason’s theorem and its generalization now state that, in all five types, the weight enumerator of a near-extremal code $C$ of length $n = Sj$ is given by

$$W(1, y) = \sum_{k=0}^{m} a_k f^{j-Rk} g^k = 1 + \sum_{k=m}^{[n/w]} A_{wk} y^k,$$

where $m = [j/R] = [n/RS]$, the $a_k$ are integers. Using the Bürmann-Lagrange theorem [18], we have

$$f^{-j} = \sum_{k=0}^{m} a_k \left( \frac{g}{f^R} \right)^k - f^{-j} \sum_{k=m}^{[n/w]} A_{wk} y^k = \sum_{s=0}^{\infty} \alpha_s \varphi^s,$$

where $\varphi = \frac{g}{f^R}$, $\alpha_0 = 1$, and for $s \geq 1$

$$\alpha_s = \frac{-j}{s!} \left[ \frac{\alpha + 2\delta y}{d\alpha y^{s-1}} \right]^{y=0}.$$

And $a_s = \alpha_s, (s = 0, 1, 2, \ldots, m - 1)$. So we have

$$\sum_{s=m}^{\infty} \alpha_s f^{j-Rs} g^s = a_m f^{j-Rm} g^m - \sum_{k=m}^{[n/w]} A_{wk} y^k. \tag{27}$$

Let $v = j - Rm, (v = 0, 1, 2, \ldots, R - 1)$. We denote the coefficient of $y^m$ as $[y^m]$. Then in the left hand side of (27), we have

$$[y^m] = \alpha_m,$$

$$[y^{m+1}] = \alpha_m \{v\alpha - w\} + \alpha_{m+1},$$

$$[y^{m+2}] = \alpha_m \left\{ v\delta - \left( \frac{v}{2} \right) \alpha^2 - wmv\alpha + \left( \frac{wm}{2} \right) \right\} + \alpha_{m+1} \{ (v-R)\alpha - w(m+1) \} + \alpha_{m+2}.$$
And in the right hand side of (27), we have
\[
\begin{align*}
[y^m] &= a_m - A_{wm}, \\
[y^{m+1}] &= a_m \{v \alpha - w m\} - A_{w(m+1)}, \\
[y^{m+2}] &= a_m \left\{ v \delta - \left(\frac{v}{2}\right) \alpha^2 - \alpha m v \alpha + \left(\frac{w m}{2}\right) \right\} - A_{w(m+2)}.
\end{align*}
\]
Therefore,
\[
\begin{align*}
A_{wm} &= a_m - \alpha_m, \\
A_{w(m+1)} &= A_{wm} \{v \alpha - w m\} - \alpha_{m+1}, \\
A_{w(m+2)} &= A_{wm} \left\{ v \delta - \left(\frac{v}{2}\right) \alpha^2 - \alpha m v \alpha + \left(\frac{w m}{2}\right) \right\} \\
&\quad - \alpha_{m+1} \left\{ (v - R) \alpha - w(m + 1) \right\} - \alpha_{m+2}.
\end{align*}
\] (28)

Since \(A_{w(m+1)} \geq 0\) and \(v \alpha - w m < 0\), from (28) we have
\[
A_{wm} \leq \frac{\alpha_{m+1}}{v \alpha - w m}.
\] (30)

By (29) and (30), we have
\[
A_{w(m+2)} \leq \frac{\alpha_{m+1}}{v \alpha - w m} \left\{ v \delta - \left(\frac{v}{2}\right) \alpha^2 - \alpha m v \alpha + \left(\frac{w m}{2}\right) \right\} \\
&\quad - \alpha_{m+1} \left\{ (v - R) \alpha - w(m + 1) \right\} - \alpha_{m+2} \\
&= -\alpha_{m+2} + \alpha_{m+1} \left\{ \frac{1}{v \alpha - w m} \left( v \delta - \left(\frac{v}{2}\right) \alpha^2 - \alpha m v \alpha + \left(\frac{w m}{2}\right) \right) \right. \\
&\quad \left. + (R - v) \alpha + w(m + 1) \right\}.
\]

Let
\[
E(m, v) = \left\{ \frac{1}{v \alpha - w m} \left( v \delta - \left(\frac{v}{2}\right) \alpha^2 - \alpha m v \alpha + \left(\frac{w m}{2}\right) \right) + (R - v) \alpha + w(m + 1) \right\}.
\]
Then, we have the following lemma.

**Lemma 7.3.** If \(\frac{\alpha_{m+2}}{\alpha_{m+1}} < E(m, v)\), then \(A_{w(m+2)} < 0\).

Let
\[
A(y, v) = (\alpha + 2 \delta y)(1 + \alpha y + \delta y^2)^{R-v-1},
\]
\[
B(y, v) = (\alpha + 2 \delta y)(1 + \alpha y + \delta y^2)^{2R-v-1},
\]

and
\[
D(m, v) = \left( w + 1 \right)^{w+1} \frac{m+1}{w} \frac{\frac{m+1}{(w+1)(m+2)-(1+\delta)(2R-v)}}{A_{w(m+2)-(1+\delta)(R-v)}}, v.
\] (31)

Now we need the following two lemmas from [20].
Lemma 7.4. $\frac{\alpha_{m+2}}{\alpha_{m+1}} < D(m, v)$.

Lemma 7.5. If $m \geq 5$, then $D(m, v) \leq D(m - 1, v)$.

Now we find the smallest $m_0$ for each type and for each $v = 0, 1, 2, \ldots, R - 1$ such that $D(m_0, v) < E(m_0, v)$ using maple software. And the results are in the following lemma.

Lemma 7.6. $D(m_0, v) < E(m_0, v)$ for

(i) binary f.s.d. even code: if $m_0 = 21, 19, 18, 18$ while $v = 0, 1, 2, 3$, respectively,

(ii) Type II self-dual code: if $m_0 = 336, 332, 331$ while $v = 0, 1, 2$, respectively,

(iii) Type III self-dual code: if $m_0 = 157, 157, 158$ while $v = 0, 1, 2$, respectively,

(iv) Type IV self-dual code: if $m_0 = 48, 46, 46$ while $v = 0, 1, 2$, respectively,

(v) f.s.d.a. odd code over $\mathbb{F}_4$: if $m_0 = 13, 13$ while $v = 0, 1$, respectively.

And we calculate $\frac{\alpha_{m+2}}{\alpha_{m+1}} - E(m, v)$ using maple software. The results are in the following Lemma.

Lemma 7.7. $\frac{\alpha_{m+2}}{\alpha_{m+1}} - E(m, v) < 0$ for

(i) binary f.s.d. even code: if $m = 11 - 20, 12 - 18, 13 - 17, 14 - 17$ while $v = 0, 1, 2, 3$, respectively,

(ii) Type II self-dual code: if $m = 315 - 335, 320 - 331, 325 - 330$ while $v = 0, 1, 2$, respectively,

(iii) Type III self-dual code: if $m = 147 - 156, 150 - 156, 154 - 157$ while $v = 0, 1, 2$, respectively,

(iv) Type IV self-dual code: if $m = 38 - 47, 41 - 45, 43 - 45$ while $v = 0, 1, 2$, respectively,

(v) f.s.d.a. odd code over $\mathbb{F}_4$: if $m = 9 - 12, 10 - 12$ while $v = 0, 1$, respectively.

To complete the proof of Theorem 7.2, we need the following lemma, where the second case improves Theorem 5.1.

Lemma 7.8. (i) binary f.s.d. even code: There is no near-extremal code with $n = 72, 80$.

(ii) f.s.d.a. odd code over $\mathbb{F}_4$: There is no near-extremal code with $n = 16$.

Proof. For (i), the results comes from [8, Theorem 6]. Now we prove (ii). Let $m = 8$. By Theorem 4.3

$$-2^{m-1} \leq a_m \leq 2^{m-1}. \quad (32)$$

And from (29)

$$A_{m+1} \leq m(\alpha_m + 2^m - 1) - \alpha_{m+1} = -960 < 0. \quad (33)$$

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This completes the proof of Theorem 7.2.

**Remark 7.9.** The coefficients of the possible weight enumerators for near-extremal f.s.d.a. odd codes over $\mathbb{F}_4$ with $n = 2i$ $(1 \leq i \leq 7)$ and all the possible values of $\alpha (= a_{n/2})$ in Theorem 4.3 are nonnegative integers.

**Remark 7.10.** We compare the idea of Zhang [20] with ours. In the first, the idea of Zhang is the following. Suppose there is an extremal self-dual code. Then one can choose $a_i$ such that

$$W(1, y) = \sum_{k=0}^{m} a_k f^{j-R_k} g^k = 1 + \sum_{k=m+1}^{[n/w]} A_w k y^k,$$

where $a_k$ $(0 \leq k \leq m)$ are uniquely determined. Then it follows that

$$A_w(m+2) < 0 \quad \text{if and only if} \quad n \geq n_0,$$

where $n_0$ is the smallest value in Theorem 1.6 in [20]. So the nonexistence of extremal self-dual codes follows. Now we explain our idea. Suppose there is a near-extremal (formally) self-dual code. Then we can choose $a_i$ such that

$$W(1, y) = \sum_{k=0}^{m} a_k f^{j-R_k} g^k = 1 + \sum_{k=m+1}^{[n/w]} A_w k y^k,$$

where $a_k$ $(0 \leq k \leq m-1)$ are uniquely determined but $a_m$ is not. Since $A_w(m+2)$ depends on $a_m,$ we do not know the sign of $A_w(m+2).$ But if we assume that $A_w(m+1) \geq 0,$ we have

$$A_w(m+2) \leq -\alpha_{m+2} + \alpha_{m+1} E(m, v),$$

and

$$-\alpha_{m+2} + \alpha_{m+1} E(m, v) < 0 \quad \text{if and only if} \quad n \geq n_0,$$

where $n_0$ is the smallest value in Lemma (7.7). So the nonexistence of near-extremal self-dual codes follows. And we can add some more conditions in the cases of the binary f.s.d. even codes with $n = 8i$ $(i = 9, 10),$ and the f.s.d.a. odd codes over $\mathbb{F}_4$ with $n = 2i$ $(i = 8).$ These cases cannot be attacked by the above method. But in these cases, we have $|a_m| \leq 2^n/4-1,$ $|a_m| \leq 2^n/2-1$ respectively. Using these values, we can deduce $A_w(m+1) < 0.$ So the nonexistence of near-extremal binary f.s.d. even codes and f.s.d.a. odd codes over $\mathbb{F}_4$ follows.

**References**


