New Extremal Self-Dual Codes of Lengths 36,38, and 58

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Abstract

We construct new extremal self-dual binary codes of lengths 36,38, and 58. We show that there are at least 14 inequivalent extremal self-dual $[36, 18, 8]$ codes and that there are at least 365 inequivalent extremal self-dual $[38, 19, 8]$ codes. For length 58 we construct 17 extremal self-dual $[58, 29, 10]$ codes whose weight enumerators were previously unknown.

I Introduction

We assume familiarity with some basic facts from coding theory. Our notation follows from [12, 13]. From now on, we consider binary linear codes. A self-dual binary code is called *doubly-even* if all codewords have weight $\equiv 0 \pmod 4$ and *singly-even* if some codeword has weight $\equiv 2 \pmod 4$. We note that self-dual binary codes exist only for even lengths. Moreover doubly-even self-dual codes exist only for lengths $n \equiv 0 \pmod 8$. Since we are interested in lengths 36,38, and 58, we are considering singly-even self-dual codes, simply self-dual codes of each length. We call self-dual codes *extremal* if it attains the highest minimum distance for that length. Conway, Pless and Sloane [4, 5] gave the complete classification of doubly-even self-dual codes of length up to 32 and singly-even self-dual codes of length up to 30. For some lengths $n \geq 34$ people have constructed extremal self-dual codes since there are many non-extremal self-dual codes [4]. Conway and Sloane [6] gave possible weight enumerators of extremal self-dual codes as well as the highest minimum distance up to length ≤ 72. For lengths 36 and 38 extremal self-dual codes have minimum distance 8 and for length 58 minimum distance 10 [6, 14].

Our purpose is to increase the number of known extremal self-dual codes of lengths 36 and 38 given by Harada [9]. More interestingly we construct 17 extremal self-dual $[58, 29, 10]$ codes whose weight enumerators were previously unknown. We also generalize Harada's construction [8] about binary self-dual codes.

II Methods

We prove the following theorem.

Theorem 1. *Let S be a subset of the set $\{1, 2, \cdots, 2n\}$ of coordinate indices such that $|S|$ is odd. Let $G_0 = (L|R) = (l_i|r_i)$ be a generator matrix(may not be in standard form) of a self-dual code $\mathcal{C}_0$ of length $2n$, where l_i and r_i are rows of L and R respectively for $1 \le i \le n$. Let $\mathbf{x} = (x_1, \cdots, x_n, x_{n+1}, \cdots, x_{2n})$ be the characteristic vector of S, i.e., $x_j := 1$ if $j \in S$ and $x_j := 0$ if $j \notin S$ for $1 \le j \le 2n$. Suppose that $y_i := (x_1, \cdots, x_n, x_{n+1}, \cdots, x_{2n}) \cdot (l_i|r_i)$ for $1 \le i \le n$. Here $\cdot$ denotes the (scalar) inner product. Then the following matrix*

$$G = \left[\begin{array}{cc|cccccc} 1 & 0 & x_1 & \cdots & x_n & x_{n+1} & \cdots & x_{2n} \\ \hline y_1 & y_1 & & & & & & \\ \vdots & \vdots & & L & & & R & \\ y_n & y_n & & & & & & \end{array}\right]$$

generates a self-dual code $\mathcal{C}$ of length $2n+2$.

Proof. It is easy to see that $\mathcal{C}$ has dimension $n+1$. It remains to show that any two rows of G are orthogonal. Note that the first row of G is orthogonal to itself. Since any two rows of G excluding the first row of G are orthogonal to each other, it suffices to show that the first row of G is orthogonal to any other rows of G. This is true since

$$\begin{aligned} &(1, 0, x_1, \cdots, x_n, x_{n+1}, \cdots, x_{2n}) \cdot (y_i, y_i, l_i|r_i) \\ &= y_i + (x_1, \cdots, x_n, x_{n+1}, \cdots, x_{2n}) \cdot (l_i|r_i) \\ &= 0 \text{ in } GF(2) \text{ for } 1 \le i \le n. \end{aligned}$$

This completes the proof. □

Choosing G_0 as $G_0 = (I_n|A)$ and $x_{n+1} = \cdots = x_{2n} = 1$ in the above theorem, we obtain Harada's construction as a corollary.

Corollary 1 (Harada [8]). *Let S be a subset of the set $\{1, 2, \cdots, n\}$ such that $|S|$ is odd if $2n \equiv 0 \pmod 4$ and $|S|$ is even if $2n \equiv 2 \pmod 4$. Let $G_0 = (I_n|A)$ be a generator matrix in standard form of a self-dual code $\mathcal{C}_0$ of length $2n$. Suppose that $x_i := 1$ if $i \in S$ and $x_i := 0$ if $i \notin S$ and that $y_i := x_i + 1$ for $1 \le i \le n$. Then the following matrix*

$$G = \left[\begin{array}{cc|cccccc} 1 & 0 & x_1 & \cdots & x_n & 1 & \cdots & 1 \\ \hline y_1 & y_1 & & & & & & \\ \vdots & \vdots & & I_n & & & A & \\ y_n & y_n & & & & & & \end{array}\right]$$

generates a self-dual code $\mathcal{C}$ of length $2n+2$.

Using Theorem 1 we can prove the following.

Theorem 2. *Any binary self-dual code $\mathcal{C}$ of length $2n$ with minimum distance $d > 2$ is obtained from some self-dual code $\mathcal{C}_0$ of length $2n-2$ (up to equivalence) by the construction in Theorem 1.*

Proof. Let G be a generator matrix of $\mathcal{C}$. Without loss of generality we may assume that $G = (I_n|A_n) = (e_i|a_i)$, where e_i and a_i are rows of I_n(the identity matrix) and A_n, respectively $(1 \le i \le n)$. It suffices to prove that there exists a vector $\mathbf{x} = (x_1, \cdots, x_{2n-2})$ and a self-dual code $\mathcal{C}_0$ of length $2n-2$ whose extended code by Theorem 1 $\mathcal{C}_1$, is a code equivalent to $\mathcal{C}$. To do that, first consider the following generator matrix of $\mathcal{C}$.

$$G' = \left[\begin{array}{c|c} e_1 + e_2 & a_1 + a_2 \\ e_2 & a_2 \\ \vdots & \vdots \\ e_n & a_n \end{array}\right]$$

We note that $e_1 + e_2 = (1, 1, 0, \cdots, 0)$ of length n. We delete the second row $(e_2|a_2)$ of G' and the first two columns of G'. Then since $d > 2$ we get a self-dual code $\mathcal{C}_0$ of length $2n-2$ with the generator matrix

$$G_0 = \left[\begin{array}{ccc|c} 0 & \cdots & 0 & a_1 + a_2 \\ & e_3 & & a_3 \\ & \vdots & & \vdots \\ & e_n & & a_n \end{array}\right].$$

Using the row $\mathbf{x} = (0, \cdots, 0|a_2)$ of length $2n-2$ and G_0 we get a generator matrix G_1 of a self-dual code $\mathcal{C}_1$ of length $2n$ by Theorem 1 since in this case S in Theorem 1 will be the set of nonzero coordinates of $\mathbf{x}$ and so $|S|$ =weight of a_2=odd. Then the given code $\mathcal{C}$ is equivalent to $\mathcal{C}_1$, as desired. This completes the proof. □

III Construction of extremal self-dual codes

A Lengths 36 and 38

Conway and Sloane [6] gave the following weight enumerators. The weight enumerator of an extremal self-dual $[36, 18, 8]$ code is either

$$W_{36,1}(y) = 1 + 225y^8 + 2016y^{10} + \cdots$$

or

$$W_{36,2}(y) = 1 + 289y^8 + 1632y^{10} + \cdots.$$

The weight enumerator of an extremal self-dual $[38, 19, 8]$ code is either

$$W_{38,1}(y) = 1 + 171y^8 + 1862y^{10} + \cdots$$

or

$$W_{38,2}(y) = 1 + 203y^8 + 1702y^{10} + \cdots.$$

They [6] gave first examples of extremal self-dual $[36, 18, 8]$ codes. The code $R2$ in [6] has weight enumerator $W_{36,1}$ with the order of automorphism group 32 and the code $D3$ in [6] has weight enumerator $W_{36,2}$ with the order of automorphism

Table I: GENERATOR MATRICES OF C_{36} and C_{34}

$$
R(C_{36}) = \begin{bmatrix}
101010011001010011 \\
110000111100100101 \\
111100101101101111 \\
101011101101111011 \\
110010010011001011 \\
111111110110010110 \\
110110100000001100 \\
101110101100000000 \\
100001100011010010 \\
110110111101011000 \\
101111010010111001 \\
010001101001000011 \\
001011010001101000 \\
000110111000010100 \\
000010011011100100 \\
000001010010101101 \\
000000101001110110 \\
000000011111111111
\end{bmatrix}, C_{34} = \left[\begin{array}{cc}
0\cdots 0 & 011010100101110110 \\
 & 111100101101101111 \\
 & 101011101101111011 \\
 & 110010010011001011 \\
 & 111111110110010110 \\
 & 110110100000001100 \\
 & 101110101100000000 \\
 & 100001100011010010 \\
I_{16} & 110110111101011000 \\
 & 101111010010111001 \\
 & 010001101001000011 \\
 & 001011010001101000 \\
 & 000110111000010100 \\
 & 000010011011100100 \\
 & 000001010010101101 \\
 & 000000101001110110 \\
 & 000000011111111111
\end{array}\right].
$$

group 34. Harada [9] gave 10 new extremal self-dual $[36, 18, 8]$ codes (10 vectors in Table 1 in [9] should be read in a reverse order [10]), proving that there are at least 12 such codes. Conway and Sloane gave first two examples of extremal self-dual $[38, 19, 8]$ codes $R3$ and $D4$ in [6] with weight enumerator $W_{38,1}$ and Harada and Kimuar gave the first example of an extremal self-dual $[38, 19, 8]$ code C_{38} in [11] with weight enumerator $W_{38,2}$. Later Harada gave 40 new extremal self-dual $[38, 19, 8]$ codes with $W_{38,1}$ and $W_{38,2}$ in [9], proving that there are at least 43 inequivalent such codes.

We start with the code C_{36} which is a self-dual $[36, 18, 6]$ code and then make a self-dual $[34, 17, 6]$ code C_{34} from C_{36} as in the proof of Theorem 2. See Table I for their generator matrices, where $R(C_{36})$ denotes the right half of the standard form $(I|A)$ of C_{36}. We obtained 2 new extremal self-dual $[36, 18, 8]$ codes using C_{34} by Theorem 1 which are not equivalent to the known 12 extremal self-dual $[36, 18, 8]$ codes in [6, 9]. For their enumeration see Table II, where vectors in the second column correspond to the characteristic vectors of the last 17 coordinates and $|\mathrm{Aut}(C)|$ in the fourth column denotes the order of the automorphism group of a corresponding code C.

Even though C_{36} is not extremal, we can still use Theorem 1. We obtained 325 inequivalent self-dual $[38, 19, 8]$ codes using C_{36} by Theorem 1, in particular Corollary 1, which are not also equivalent to Harada's 40 extremal self-dual $[38, 19, 8]$ codes in [9]. The equivalence was checked by Magma [1]. For their enumeration see Table V, VI, and VII. Here the first column denotes codes C numbered as first, second, etc., the second column has 6 vectors separated by ';' and those vectors are written in octal using $0 = (000), 1 = (001), \cdots, 7 = (111)$ and correspond to the characteristic vectors of the first 18 coordinates, the rest being $1, \cdots, 1$(18 1's) and $|\mathrm{Aut}(C)|$ in the third column denotes the order of the automorphism group of a corresponding code C.

We summarize our results as follows.

Theorem 3. *There are at least* $12+2 = 14$ *inequivalent extremal self-dual* $[36, 18, 8]$ *codes. There are at least* $40 + 325 = 365$ *inequivalent extremal self-dual* $[38, 19, 8]$

Table II: NEW EXTREMAL SELF-DUAL CODES OF LENGTH 36

codes C	vectors $\mathbf{x} = (1, \cdots, 1, x_{18}, \cdots, x_{34})$	enumerator	$\lvert\mathrm{Aut}(C)\rvert$	using
$C_{36,11}$	1 1 0 1 0 1 1 1 0 0 1 0 0 1 0 0 0	$W_{36,1}$	6	C_{34}
$C_{36,12}$	0 0 0 1 0 0 1 0 1 0 0 1 1 0 1 1 1	$W_{36,2}$	12	C_{34}

codes.

B Length 58

Conway and Sloane [6] showed that the weight enumerator of an extremal self-dual binary $[58, 29, 10]$ code is either

$$W_{58,1}(y) = 1 + (165 - 2\gamma)y^{10} + (5078 + 2\gamma)y^{12} + \cdots$$

or

$$W_{58,2}(y) = 1 + (319 - 24\beta - 2\gamma)y^{10} + (3132 + 152\beta + 2\gamma)y^{12} + \cdots,$$

where β and γ are undetermined parameters.

Tsai [15] constructed a self-dual code with weight enumerator $W_{58,1}$ with $\gamma = 55$. For the second weight enumerator $W_{58,2}$ several authors constructed the following codes. Conway and Sloane [6] gave examples of codes with weight enumerator $W_{58,2}$ for $\beta = 0$ and $\gamma \in \{0, 58\}$. Harada and Kimura [11] constructed such codes for $\beta = 0$ and $\gamma \in \{52\} \cup \{2m | 30 \leq m \leq 42\}$. Buyuklivea and Boukliev [2, 3] found self-dual codes for

$$\begin{aligned} &\beta = 0 \text{ and } \gamma \in \{32, 36, 40, 48, 50, 54, 56, 58\} \cup \{2m | 43 \leq m \leq 56\}, \\ &\beta = 1 \text{ and } \gamma \in \{62, 64\} \cup \{2m | 34 \leq m \leq 50\}, \\ &\beta = 2 \text{ and } \gamma \in \{4m | 11 \leq m \leq 21\}. \end{aligned}$$

Tsai and Jiang [16] obtained such codes for

$$\begin{aligned} &\beta = 0 \text{ and } \gamma \in \{114, 116, 118, 120, 122\}, \\ &\beta = 2 \text{ and } \gamma \in \{62, 70, 74, 78, 82, 86, 88\}. \end{aligned}$$

Using Theorem 1, in particular Corollary 1 and two generator matrices of self-dual $[56, 28, 10]$ codes C_{56}^1, C_{56}^2 in Table III, we obtain 17 extremal self-dual $[58, 29, 10]$ codes(see Table IV) with weight enumerator $W_{58,2}$ for

$$\begin{aligned} &\beta = 0 \text{ and } \gamma \in \{42, 46\}, \\ &\beta = 1 \text{ and } \gamma \in \{2m | 21 \leq m \leq 30\} \cup \{66\}, \\ &\beta = 2 \text{ and } \gamma \in \{50, 54, 58, 66\}. \end{aligned}$$

In Table III, $R(C_{56}^i)(i = 1, 2)$ denotes the right half of the standard form $(I|A)$ of $C_{56}^i(i = 1, 2)$. In Table IV, vectors in the second column correspond to the characteristic vectors of the first 23 coordinates, the rest being $0, 0, 0, 0, 0, 1, \cdots, 1$(28 1's).

Table III: GENERATOR MATRICES OF C_{56}^1 and C_{56}^2

$$R(C_{56}^1) = \begin{bmatrix} 1111011110101001011000100111 \\ 0001011100110111101111011001 \\ 0011000001001111011011111111 \\ 1111011100001001111110000001 \\ 0000010001111011000000111111 \\ 0000110110001101000100011111 \\ 1110100111101000110001110001 \\ 1110101110010101010000111001 \\ 1110101010101011100000011101 \\ 1110101000110100111000001111 \\ 1110101001111011010100000110 \\ 0100101011000010010101111100 \\ 1000101001001111011001000000 \\ 0111101011011000010011011111 \\ 0011001011011100101101101111 \\ 0001011011011110110010110111 \\ 0000010011011111111101011011 \\ 0000110111011111011010101101 \\ 0000100101011111001001010110 \\ 0100101100011111000000101011 \\ 1100101010100001110011101011 \\ 1111101000110001110001110100 \\ 1010001001111001110000111011 \\ 1100111001011101110000011100 \\ 0101100011010001000111110001 \\ 0010001111011000000111111000 \\ 1011111011000010010000000010 \\ 0111010100100000111000000000 \end{bmatrix}, \quad R(C_{56}^2) = \begin{bmatrix} 1011100000100111011011111001 \\ 0100110101101110001111111001 \\ 1100010101101111100011111111 \\ 0001011110010111001001111111 \\ 1111000101011011111000111111 \\ 1111100010101101111100011111 \\ 0000100101110110000110001111 \\ 0000101100000101110011100011 1 \\ 0000101000110101010111100011 \\ 1111111110001010110111110001 \\ 1111111111000101011011111000 \\ 1011111111100010101101111100 \\ 1001111111110001010110111110 \\ 1000111111111000101011011111 \\ 1100011111111100010101101111 \\ 1110001111111110001010110111 \\ 1111000111111111000101011011 \\ 0000110111011111011010101101 \\ 1111110001111111110001010110 \\ 1011111000111111111000101011 \\ 0010101000111111000100010101 \\ 0001101010101111000110001010 \\ 1011011111000111111111000101 \\ 0010111011000011000111100010 \\ 1010110111110001111111110001 \\ 1101011011111000111111111000 \\ 0101111001011100100111111100 \\ 0110000010011110110111111110 \end{bmatrix}.$$

Table IV: NEW EXTREMAL SELF-DUAL CODES OF LENGTH 58

codes C	vectors $\mathbf{x} = (x_1, \cdots, x_{23}, 0, 0, 0, 0, 0, 1, \cdots, 1)$	β	γ	using
$C_{58,1}$	0 1 0 0 0 1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 0 1 0	0	42	C_{56}^2
$C_{58,2}$	0 0 0 0 0 1 1 0 0 0 0 0 0 1 0 0 0 1 0 0 1 0 0	0	46	C_{56}^1
$C_{58,3}$	1 0 0 1 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0	1	42	C_{56}^1
$C_{58,4}$	1 0 1 0 1 0 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0	1	44	C_{56}^2
$C_{58,5}$	1 0 0 1 1 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0	1	46	C_{56}^1
$C_{58,6}$	1 0 1 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1	1	48	C_{56}^1
$C_{58,7}$	1 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0	1	50	C_{56}^1
$C_{58,8}$	1 1 0 0 1 0 1 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0	1	52	C_{56}^1
$C_{58,9}$	1 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 1 0 1 0	1	54	C_{56}^1
$C_{58,10}$	1 1 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0	1	56	C_{56}^1
$C_{58,11}$	1 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0	1	58	C_{56}^1
$C_{58,12}$	1 0 1 0 1 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0	1	60	C_{56}^1
$C_{58,13}$	1 1 1 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0	1	66	C_{56}^1
$C_{58,14}$	1 1 0 0 0 0 0 0 0 0 0 1 0 0 1 1 0 0 0 0 0 0 0	2	50	C_{56}^1
$C_{58,15}$	1 1 0 0 0 0 0 0 0 0 1 0 0 0 0 1 0 0 0 1 0 0 0	2	54	C_{56}^1
$C_{58,16}$	1 1 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0 0 1 0 0 0	2	58	C_{56}^1
$C_{58,17}$	1 0 0 0 0 1 0 0 0 1 0 1 0 0 0 0 0 0 1 0 0 0 0	2	66	C_{56}^1

Table V: NEW EXTREMAL SELF-DUAL CODES OF LENGTH 38 with $W_{38,1}$

codes C	vectors $\mathbf{x} = (x_1, \cdots, x_{18}, 1, \cdots, 1)$	$\lvert\mathrm{Aut}(C)\rvert$	using
$1 - 6$	273274; 472536; 143755; 715174; 702657; 672435	1; 1; 1; 1; 1; 1	C_{36}
$7 - 12$	361574; 703574; 263574; 652635; 272635; 746135	1; 1; 1; 2; 1; 1	C_{36}
$13 - 18$	666135; 467135; 340757; 760335; 364335; 312735	1; 1; 1; 1; 1; 1	C_{36}
$19 - 24$	433437; 346237; 077075; 431637; 346137; 773302	1; 1; 1; 2; 1; 1	C_{36}
$25 - 30$	433175; 250737; 337642; 614277; 773320; 676641	1; 1; 1; 1; 1; 1	C_{36}
$31 - 36$	537641; 773222; 766341; 374741; 771660; 337322	1; 1; 1; 1; 1; 1	C_{36}
$37 - 42$	727360; 770543; 374543; 732543; 375462; 175543	1; 1; 1; 1; 1; 1	C_{36}
$43 - 48$	173543; 517543; 457543; 371662; 171743; 353662	1; 1; 1; 1; 1; 1	C_{36}
$49 - 54$	633662; 671562; 772223; 376223; 575223; 477223	1; 1; 1; 1; 1; 1	C_{36}
$55 - 60$	766261; 656661; 555661; 535661; 533661; 674323	1; 1; 1; 2; 1; 1	C_{36}
$61 - 66$	555323; 175323; 533323; 354761; 334761; 771512	1; 1; 1; 1; 1; 1	C_{36}
$67 - 72$	455761; 553463; 773150; 763312; 565263; 353712	1; 1; 1; 1; 1; 2	C_{36}
$73 - 78$	155663; 135663; 527163; 467163; 664363; 523363	1; 1; 1; 1; 1; 1	C_{36}
$79 - 84$	676511; 766311; 573413; 157711; 572352; 476352	1; 1; 1; 1; 1; 1	C_{36}
$85 - 90$	273352; 567213; 571613; 535613; 553613; 517613	1; 1; 1; 1; 1; 2	C_{36}
$91 - 96$	437613; 137613; 371730; 672513; 771432; 565651	1; 1; 1; 1; 1; 1	C_{36}
$97 - 102$	517513; 565313; 677070; 371632; 433713; 636351	1; 1; 1; 1; 1; 1	C_{36}
$103 - 108$	637270; 364751; 545751; 525751; 267332; 472732	1; 1; 1; 1; 1; 1	C_{36}

Table VI: NEW EXTREMAL SELF-DUAL CODES OF LENGTH 38 with $W_{38,1}$

codes C	vectors $\mathbf{x} = (x_1, \cdots, x_{18}, 1, \cdots, 1)$	$\lvert\mathrm{Aut}(C)\rvert$	using
$109-114$	465653; 165653; 067653; 273370; 077153; 656631	1; 1; 2; 1; 1; 1	C_{36}
$115-120$	165553; 256353; 075353; 433353; 073353; 740753	1; 1; 1; 1; 1; 2	C_{36}
$121-126$	674531; 344753; 143753; 275172; 516372; 773304	1; 2; 1; 1; 1; 1	C_{36}
$127-132$	737304; 467133; 765704; 561671; 465671; 773206	1; 1; 1; 1; 1; 1	C_{36}
$133-138$	077171; 350733; 352371; 433371; 720771; 344771	1; 1; 1; 1; 1; 1	C_{36}
$139-144$	776205; 257744; 602773; 736305; 776007; 574546	1; 1; 2; 1; 1; 1	C_{36}
$145-150$	762705; 751546; 715546; 355546; 335546; 317546	2; 1; 1; 1; 1; 2	C_{36}
$151-156$	707346; 774445; 676445; 733324; 673324; 725724	1; 1; 1; 1; 1; 1	C_{36}
$157-162$	665724; 673226; 535307; 636545; 735464; 375464	1; 1; 1; 1; 1; 1	C_{36}
$163-168$	665626; 663626; 535545; 173545; 371664; 276447	1; 1; 1; 1; 1; 1	C_{36}
$169-174$	371564; 275564; 171647; 712547; 376027; 535325	1; 1; 1; 1; 1; 1	C_{36}
$175-180$	475325; 477027; 651566; 331566; 735614; 674227	1; 1; 2; 1; 1; 1	C_{36}
$181-186$	703366; 134766; 760627; 735514; 717514; 356127	1; 1; 1; 1; 1; 2	C_{36}
$187-192$	336127; 771416; 335714; 617714; 666165; 763216	1; 1; 1; 1; 1; 1	C_{36}
$193-198$	712565; 371616; 335616; 073565; 237616; 706365	1; 1; 1; 1; 1; 1	C_{36}
$199-204$	646365; 346365; 346267; 267316; 252667; 676415	1; 1; 1; 1; 1; 1	C_{36}
$205-210$	431667; 656615; 553615; 161367; 610767; 636515	1; 1; 1; 1; 1; 1	C_{36}
$211-216$	623656; 746315; 564556; 723334; 272517; 277074	1; 1; 1; 3; 1; 1	C_{36}
$217-222$	327236; 623066; 550466; 154466; 644133; 130566	1; 1; 1; 1; 1; 1	C_{36}
$223-228$	211566; 110766; 710073; 576440; 315416; 703216	1; 1; 2; 1; 1; 1	C_{36}
$229-234$	564116; 134516; 560316; 534056; 174056; 654207	1; 1; 1; 1; 1; 1	C_{36}
$235-240$	444656; 363620; 262507; 640707; 722047; 512436	1; 1; 1; 1; 1; 1	C_{36}
$241-246$	252447; 327260; 161247; 631660; 503427; 700627	1; 1; 1; 1; 1; 1	C_{36}
$247-252$	653610; 176510; 242527; 403527; 606067; 411467	1; 1; 1; 2; 1; 1	C_{36}
$253-258$	700267; 105267; 210667; 174350; 524750; 136630	1; 1; 1; 1; 1; 1	C_{36}
$259-264$	651630; 351630; 311730; 353070; 571221; 163621	1; 1; 1; 1; 1; 1	C_{36}
$265-270$	127621; 652321; 163261; 650561; 672411; 533411	1; 1; 1; 1; 2; 1	C_{36}
$271-276$	360651; 324651; 702651; 265624; 521551; 232351	1; 1; 1; 1; 2; 1	C_{36}
$277-282$	670431; 314631; 516614; 151171; 121571; 623314	1; 1; 1; 1; 1; 1	C_{36}
$283-288$	633054; 730445; 654445; 165145; 031565; 574302	1; 1; 1; 1; 1; 1	C_{36}
$289-294$	076642; 630435; 431435; 550662; 170662; 311562	2; 1; 1; 1; 1; 1	C_{36}
$295-300$	141475; 554252; 163243; 740623; 506706; 454646	1; 1; 1; 1; 1; 1	C_{36}
$301-303$	033613; 054746; 225620	1; 1; 1	C_{36}

Table VII: NEW EXTREMAL SELF-DUAL CODES OF LENGTH 38 with $W_{38,2}$

codes C	vectors $\mathbf{x} = (x_1, \cdots, x_{18}, 1, \cdots, 1)$	$\lvert\mathrm{Aut}(C)\rvert$	using
$304-309$	352355; 315374; 255374; 255276; 515535; 523335	1; 1; 1; 1; 1; 1	C_{36}
$310-315$	462576; 540776; 107575; 103775; 333742; 136762	1; 1; 1; 1; 2; 1	C_{36}
$316-321$	765452; 367452; 663472; 407753; 342673; 367524	1; 1; 1; 1; 1; 1	C_{36}
$322-325$	431127; 243554; 076542; 250355	1; 1; 1; 1	C_{36}

ACKNOWLEDGMENT

The author wishes to thank his advisor Prof. Vera Pless for helpful comments. He also wishes to thank Dr. Masaaki Harada for useful discussions.

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