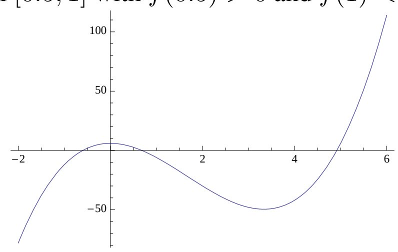
MATH 205: EXTRA CREDIT SOLUTIONS

SPRING 2009

Extra Credit 1 (February 2)**.** Use the bisection method to find the "middle" solution of $3x^3 - 15x^2 + 6 = 0$ to five decimal places. No credit will be given without documentation of your work.

Solution. Let $f(x) = 3x^3 - 15x^2 + 6$. From a graph of the function, we see that the middle solution lies in the interval $[0.5, 1]$ with $f(0.5) > 0$ and $f(1) < 0$.



Since $f(0.75) \approx -1.17 < 0$, we see the solution must lie in the interval $[0.5, 0.75]$. Continuing in this way, using midpoints of intervals to determine in which half the solution resides, we generate a sequence of intervals $[a_n, b_n]$.

a_n	b_n	$m_n = \frac{a_n+b_n}{2}$	$f(a_n)$	$f(b_n)$	$f(m_n)$
0.50000	0.75000	0.62500	2.62500	−1.17188	0.87305
0.62500	0.75000	0.68750	0.87305	−1.17188	−0.11499
0.62500	0.68750	0.65625	0.87305	−0.11499	0.38791
0.65625	0.68750	0.67188	0.38791	−0.11499	0.13865
0.67188	0.68750	0.67969	0.13865	−0.11499	0.01237
0.67969	0.68750	0.68359	0.01237	−0.11499	−0.05118
0.67969	0.68359	0.68164	0.01237	−0.05118	−0.01937
0.67969	0.68164	0.68066	0.01237	−0.01937	−0.00349
0.67969	0.68066	0.68018	0.01237	−0.00349	0.00444
0.68018	0.68066	0.68042	0.00444	−0.00349	0.00048
0.68042	0.68066	0.68054	0.00048	−0.00349	−0.00151
0.68042	0.68054	0.68048	0.00048	−0.00151	−0.00052
0.68042	0.68048	0.68045	0.00048	−0.00052	−0.00002
0.68042	0.68045	0.68044	0.00048	−0.00002	0.00023
0.68044	0.68045	0.68044	0.00023	−0.00002	0.00010
0.68044	0.68045	0.68045	0.00010	−0.00002	0.00004
0.68045	0.68045	0.68045	0.00004	−0.00002	0.00001

To five decimal places, the solution is seen to be 0.68045. (This table was generated by the spreadsheet from Open Office.) □

Extra Credit 2 (February 9)**.** Suppose $f : [0, 1] \to [0, 1]$ is continuous. Must the equation $f(x) = x$ have a solution? If not, draw a counterexample. If so, explain why.

Solution. If $g(x) = f(x) - x$, then g is continuous on $[0, 1]$ because it is the sum of two continuous functions. If $f(0) = 0$ or $f(1) = 1$, then $f(x) = x$ does have a solution in $[0, 1]$. Otherwise, it must be the case that $f(0) > 0$ and $f(1) < 1$; i.e., $g(0) = f(0) - 0 > 0$ and $g(1) = f(1) - 1 < 0$. Since g is continuous, the Intermediate Value Theorem guarantees there is an $x_0 \in (0, 1)$ with $g(x_0) = 0$. From the definition of g, we see $f(x_0) - x_0 = 0$, or $f(x_0) = x_0$. □

Extra Credit 3 (February 23). Let $f(x) = \ln x$ and $q(x) = ax^2 + bx + c$. Find a, b and c so $f(1) = q(1)$, $f'(1) = q'(1)$ and $f''(1) = q''(1)$.

Solution. Evidently,

$$f(x) = \ln x,\ f'(x) = \frac{1}{x},\ f''(x) = \frac{-1}{x^2} \implies f(1) = 0,\ f'(1) = 1,\ f''(1) = -1 \tag{1}$$

and

$$\begin{aligned} & q(x) = ax^2 + bx + c,\ q'(x) = 2ax + b,\ q''(x) = 2a \\ & \implies q(1) = a + b + c,\ q'(1) = 2a + b,\ q''(1) = 2a. \end{aligned} \tag{2}$$

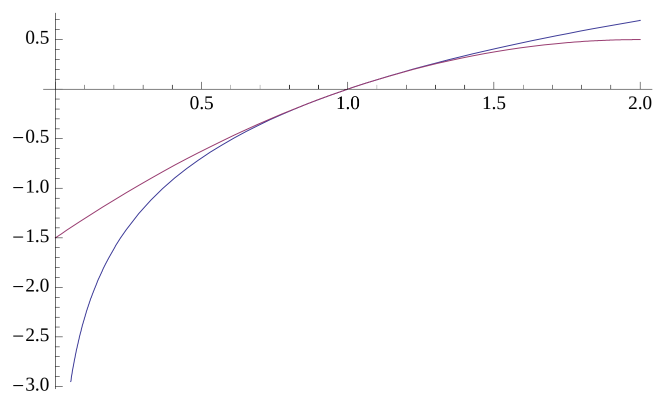
Equating (1) and (2) gives the following system of equations

$$\begin{aligned} a + b + c &= 0 \\ 2a + b &= 1 \\ 2a &= -1 \end{aligned}$$

This is easily solved to see $a = -1/2$, $b = 2$ and $c = -3/2$, giving

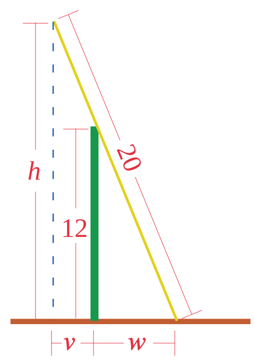
$$q(x) = -\frac{1}{2}x^2 + 2x - \frac{3}{2}.$$

The function q is a parabola tangent to the graph of f at $(1, 0)$. They're both plotted below.



□

Extra Credit 4 (March 9)**.** A 20 foot ladder leans against a 12 foot high wall with the top of the ladder projecting over the wall. Its bottom is pulled away from the wall at 5 feet per second. How fast is the height of the top of the ladder decreasing when the top of the ladder reaches the top of the wall?



Solution. Referrring to the diagram above, we know $dw/dt = 5$ and from the Pythagorean Theorem that

$$h^2 + (v + w)^2 = 400 \tag{3}$$

and from similar triangles

$$\frac{v + w}{h} = \frac{w}{12} \implies v = \frac{hw}{12} - w. \tag{4}$$

We want to determine dh/dt when $v = 0$.

Eliminate v by substituting (4) into (3) and then simplify:

$$h^2 \left(1 + \frac{w^2}{144}\right) = 400.$$

Differentiate both sides and simplify:

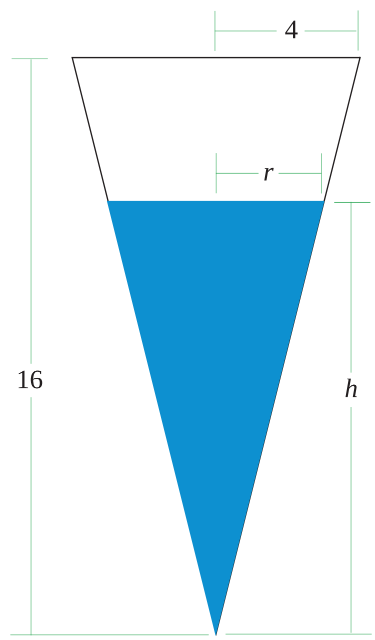
$$\frac{dh}{dt}\left(1 + \frac{w^2}{144}\right) + \frac{h\,w}{144}\frac{dw}{dt} = 0$$

When $v = 0$ we see $h = 12$ and (3) gives $w = 16$, so

$$\left.\frac{dh}{dt}\right|_{v=0} = -\frac{12}{5} \text{ ft/s}.$$

□

Extra Credit 5 (March 24)**.** A water tank has the shape of a cone 16 ft high with a radius of 4 ft at the top. Water is being pumped in at 10 ft^3 per minute and is being removed from the bottom. At the time the water is 12 ft deep, the depth is increasing at 4 inches per minute. How fast is the water being removed?



Solution. Let h and r be as in the figure, V the volume of water in the tank and a be the rate at which water is being drained. Using similar triangles from the figure

$$\frac{4}{16} = \frac{r}{h} \implies r = \frac{h}{4}. \tag{5}$$

The volume of water in the tank is $V = \frac{1}{3}\pi r^2 h$. Using (5) to eliminate r, this becomes

$$V = \frac{\pi}{48}h^3 \implies \frac{dV}{dt} = \frac{\pi}{16}h^2\frac{dh}{dt}.$$

Since the rate of change of the volume of water in the tank is the rate at which it is entering minus the rate at which it is leaving, $dV/dt = 10 - a$. Combining these two expressions for dV/dt gives

$$\frac{\pi}{16}h^2\frac{dh}{dt} = 10 - a \implies a = 10 - \frac{\pi}{16}h^2\frac{dh}{dt}.$$

At the instant when $h = 12$ and $dh/dt = 1/3$,

$$a = 10 - \frac{\pi}{16}(12)^2\frac{1}{3} = 10 - 3\pi \text{ ft}^3\text{/min}.$$

(This is about 0.575 ft^3/min.) □

Extra Credit 6 (March 31)**.** For n a positive integer, graph $y = x^n/e^x$. Indicate all critical numbers, inflection points and asymptotes. (Make sure you tell how the shape of the graph depends on n.)

Solution. First note that

$$\lim_{x\to-\infty} f(x) = \begin{cases} \infty, & n \text{ even} \\ -\infty, & n \text{ odd} \end{cases} \quad \text{and} \quad \lim_{x\to\infty} f(x) = 0,$$

so f has the positive x-axis as a horizontal asymptote.

When $n = 1$, $f(x) = x/e^x$, $f'(x) = (x-1)/e^x$ and $f''(x) = (x-2)/e^x$ shows that f has $x = 1$ as a critical number and $x = 2$ as a possible inflection point. See Figure 1.

When $n = 2$, $f(x) = x^2/e^x$, $f'(x) = x(x-2)/e^x$ and $f''(x) = ((x-2)^2 - 2)/e^x$ shows f has $x = 0, 2$ as critical numbers and $x = 2 \pm \sqrt{2}$ as possible inflection points. See Figure 2.

When $n > 2$, $f(x) = x^n/e^x$, $f'(x) = x^{n-1}(n-x)/e^x$ and $f''(x) = x^{n-2}((x-n)^2 - n)/e^x$ shows f has critical numbers at $x = 0, n$ and possible inflection points at $x = 0, n \pm \sqrt{n}$. The graphs come in two flavors. When n is even, there is no inflection, but there is a minimum $x = 0$ and when n is odd, there is an inflection at $x = 0$, which is not where an extreme occurs. These are shown in Figures 3 and 4, respectively.

□

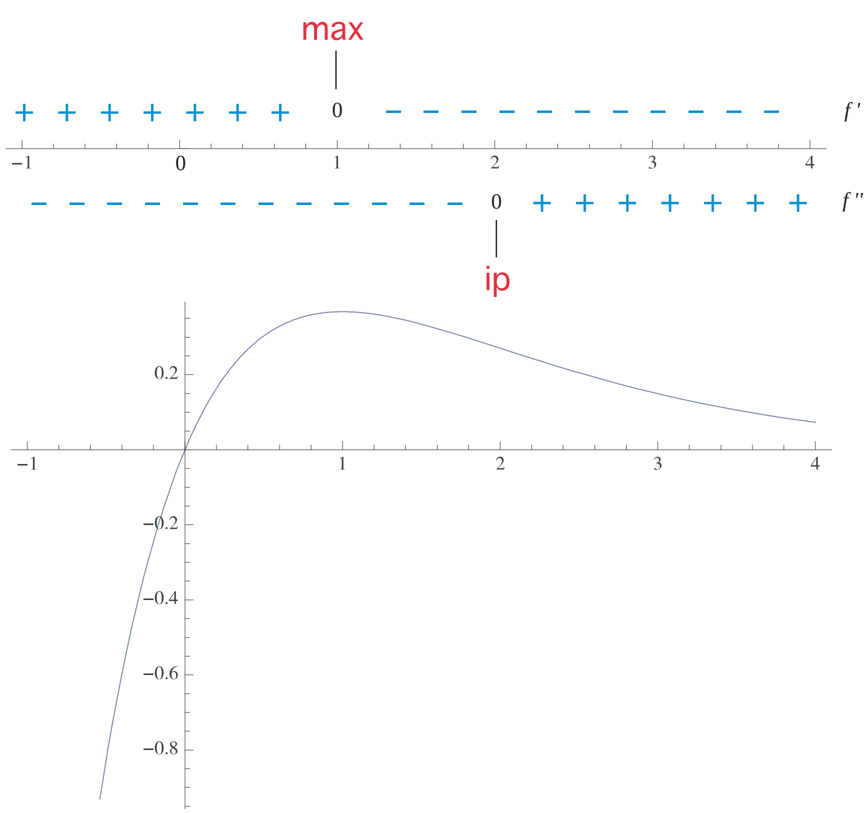


FIGURE 1. This is a plot of $f(x) = x/e^x$.

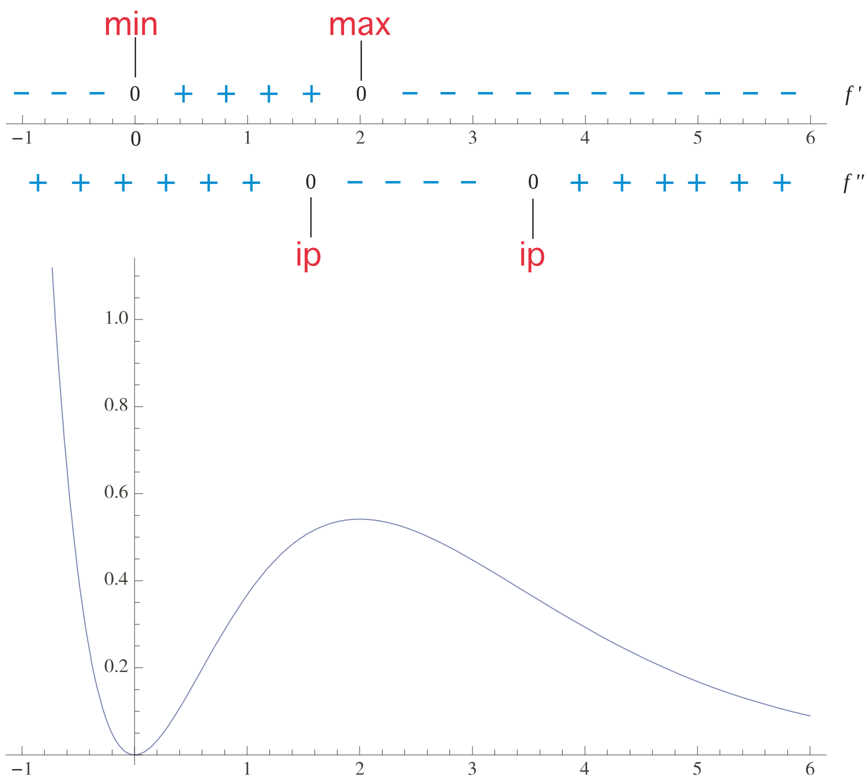
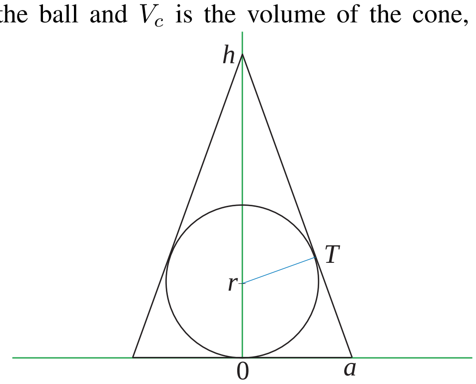


FIGURE 2. This is a plot of $f(x) = x^2/e^x$.

Extra Credit 7 (April 7)**.** A ball of radius r is placed into a right circular cone. If V_b is the volume of the ball and V_c is the volume of the cone, then minimize V_c/V_b.



Solution. The diagram is a cross section of the cone and the ball, where a is the base radius of the cone and h is its height. Therefore,

$$V_b = \frac{4\pi}{3} r^3 \text{ and } V_c = \frac{\pi}{3} a^2 h.$$

Since the triangles with vertices at $O, (a, 0), (0, h)$ and $T, (0, r), (0, h)$ are similar right triangles, we see

$$\frac{\sqrt{h^2 + a^2}}{a} = \frac{h - r}{r} \implies a^2 = \frac{r^2 h}{h - 2r}.$$

Letting $\rho = V_c / V_b$ and substituting in the expression for a^2 yields

$$\rho = \frac{\frac{\pi}{3} a^2 h}{\frac{4\pi}{3} r^3} = \frac{a^2 h}{4r^3} = \frac{\frac{r^2 h}{h-2r} h}{4r^3} = \frac{1}{4r} \frac{h^2}{h - 2r},$$

□

where $h > 2r$. A simple calculation gives

$$\frac{d\rho}{dh} = \frac{1}{4r} \frac{h(h - 4r)}{(h - 2r)^2}$$

with the only critical number in the range $h > 2r$ at $h = 4r$. The first derivative test shows ρ has a minimum value of 2 when $h = 4r$.

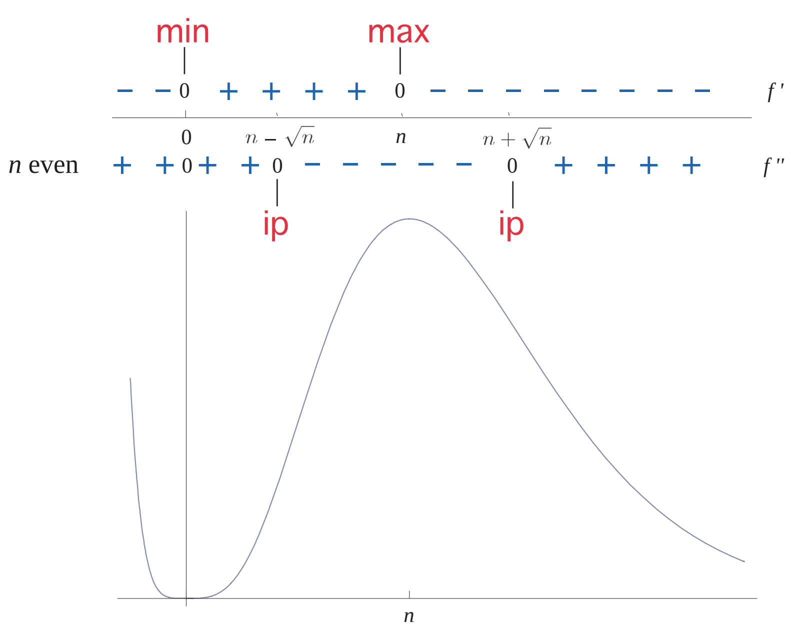


FIGURE 3. When $n > 2$ and n is even, there is no inflection at $x = 0$.

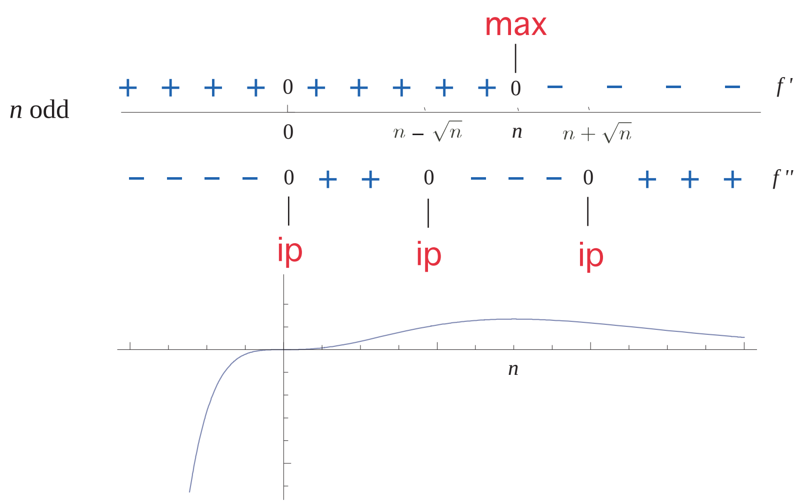


FIGURE 4. When $n > 2$ and n is odd, there is an inflection at $x = 0$.

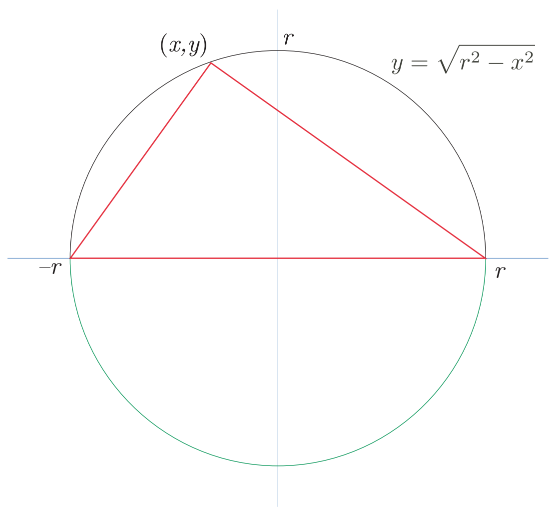


FIGURE 5. The triangle of Extra Credit 8.

Extra Credit 8 (April 14)**.** A fixed circle lies in the plane. A triangle is drawn inside the circle with all three vertices on the circle and two of the vertices at the ends of a diameter. Where should the third vertex lie to maximize the perimeter of the triangle?

Solution. Suppose the circle has radius r and is situated as in Figure 5. Then, we know

$$y = \sqrt{r^2 - x^2} \tag{6}$$

and the perimeter is

$$p = 2r + \sqrt{(x+1)^2 + y^2} + \sqrt{(x-1)^2 + y^2}. \tag{7}$$

Substitute (6) into (7) to eliminate y:

$$\begin{aligned} p &= 2r + \sqrt{(x+1)^2 + r^2 - x^2} + \sqrt{(x+1)^2 + r^2 - x^2} \\ &= 2r + \sqrt{1 + r^2 + 2x} + \sqrt{1 + r^2 - 2x} \end{aligned}$$

We must find the maximum value of p for $0 \le x \le r$. To do so, find the derivative.

$$\begin{aligned} \frac{dp}{dx} &= \frac{1}{\sqrt{1 + r^2 + 2x}} - \frac{1}{\sqrt{1 + r^2 - 2x}} \\ &= \frac{\sqrt{1 + r^2 - 2x} - \sqrt{1 + r^2 + 2x}}{\sqrt{(1 + r^2 + 2x)(1 + r^2 - 2x)}} \end{aligned}$$

The derivative exists everywhere for $0 \le x \le r$, so we search for critical numbers by finding where the numerator is 0.

$$\begin{aligned} \sqrt{1 + r^2 - 2x} - \sqrt{1 + r^2 + 2x} &= 0 \\ \sqrt{1 + r^2 - 2x} &= \sqrt{1 + r^2 + 2x} \\ 1 + r^2 - 2x = 1 + r^2 + 2xx &= 0 \end{aligned}$$

This shows the isosceles triangle with vertex at $(0, r)$ in the figure has the largest perimeter.

$\square$