

MATH 205: Sample Final Exam Problem Solutions

Here are solutions to the sample final exam problems. Please let me know about mistakes and typos.

1. Differentiate the following functions.

(a) $f(x) = (x^2 + 1)^2$
 $f'(x) = 4x(x^2 + 1)$

(b) $g(t) = \frac{\sin t}{1 + \cos t}$
 $g'(t) = \frac{\cos t(1 + \cos t) - \sin t(-\sin t)}{(1 + \cos t)^2}$
 $= \frac{1 + \cos^2 t + \sin^2 t}{(1 + \cos t)^2}$
 $= \frac{1}{(1 + \cos t)^2}$

(c) $h(w) = x^w$
 $h(w) = e^{w \ln x}$
 $h'(w) = (\ln x)e^{w \ln x} = x^w \ln x$

(d) $x + xy = e^{xy}$ (Find dy/dx .)
 $1 + y + xy' = (y + xy')e^{xy}$
 $xy'(1 - e^{xy}) = ye^{xy} - y - 1$
 $y' = \frac{ye^{xy} - y - 1}{x(1 - e^{xy})}$

(e) $e(t) = \ln \sin t^2$
 $e'(t) = \frac{2t \cos t^2}{\sin t^2} = 2t \cot t^2$

(f) $p(t) = \log_{3t} t^3$
 $p(t) = 3 \log_{3t} t = \frac{3 \ln t}{\ln 3t}$
 $p'(t) = \frac{\frac{3 \ln 3t}{t} - \frac{9 \ln t}{3t}}{(\ln 3t)^2}$
 $= \frac{3 \ln 3}{t(\ln 3t)^2}$

2. If $y = 10^x$, then find $y''(0)$.

$$y = e^{x \ln 10}$$

$$y' = (\ln 10)e^{x \ln 10}$$

$$y'' = (\ln 10)^2 e^{x \ln 10}$$

$$y''(0) = (\ln 10)^2$$

3. Find an equation of the tangent line to $y = x^3 - 9x + 1$ at $x = 0$. Use the tangent line to estimate the value of y when $x = 0.1$.

SOLUTION. Find the slope of the tangent line at $(0, 1)$: $y' = 3x^2 - 9 \implies y'(0) = -9$. From the point-slope formula, the tangent line is

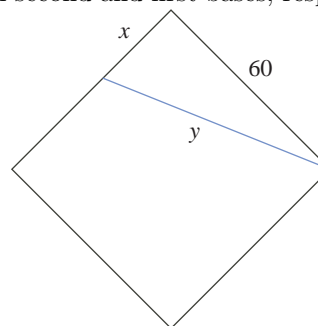
$$y - 1 = -9(x - 0) \implies y = -9x + 1.$$

The estimate is the y -coordinate on the tangent line when $x = 0.1$

$$y(0.1) = -9(0.1) + 1 = 0.1.$$

4. A softball diamond has the shape of a square with sides 60 feet long. If a player is running from second to third at 24 ft/sec, then how fast is her distance from first base changing when she is 20 feet past second base?

SOLUTION. Let x and y be the distances of the runner from second and first bases, respectively.



Know: $\frac{dx}{dt} = 24$

$$x^2 + 60^2 = y^2 \tag{1}$$

Want: dy/dt when $x = 20$ and $y = 20\sqrt{10}$.

Differentiate (1)

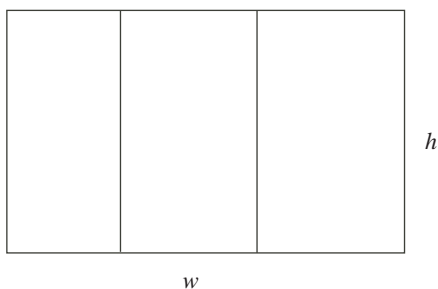
$$2x \frac{dx}{dt} = 2y \frac{dy}{dt} \implies \frac{dy}{dt} = \frac{24x}{y}$$

so that

$$\left. \frac{dy}{dt} \right|_{x=20} = \frac{24 \times 20}{20\sqrt{10}} = \frac{24}{\sqrt{10}} \text{ ft/s.}$$

5. A field is to be made into the shape of a rectangle. This field is to be further subdivided into three smaller rectangles by placing two fences parallel to one of the sides. If 1000 m of fencing are to be used, then what is the area of the largest such field?

SOLUTION. Let h and w be the outer dimensions of the field and A be its area.



Know: $2w + 4h = 1000 \implies w = 500 - 2h$

$$A = hw \implies A = h(500 - 2h) = 500h - 2h^2$$

Want: The maximum of A for $0 \leq h \leq 250$.

$$A'(h) = 500 - 4h$$

The only critical number occurs when $h = 125$. This is clearly where the maximum area occurs because $A''(h) = -4 < 0$ everywhere. (Or, just note the graph of $A(h)$ is a parabola opening downward.) Finally, $A(125) = 31250 \text{ m}^2$.

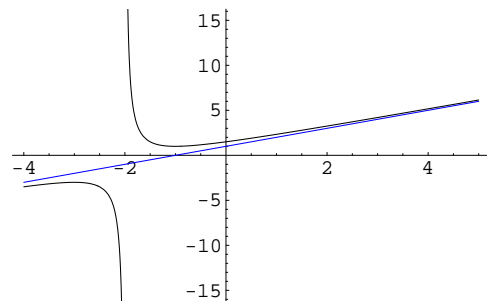
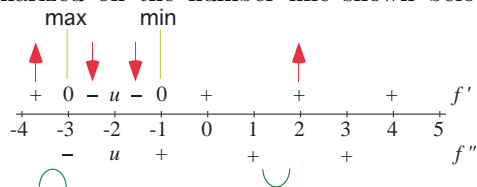
6. Graph $f(x) = \frac{1}{x+2} + x + 1$, showing all important details such as critical points, concavity and

asymptotes. On what intervals is this function decreasing?

SOLUTION. First, note the function has a vertical asymptote at $x = -2$ and $y = x + 1$ is an oblique asymptote.

$$f'(x) = \frac{-1}{(x+2)^2} + 1, \quad f''(x) = \frac{2}{(x+2)^3}$$

From the first derivative we get critical numbers at $x = -3, -2, -1$ and the second derivative gives a possible inflection point at $x = -2$. The sign distribution is summarized on the number line shown below.



On the graph given above, the blue line is the oblique asymptote.

7. Find the maximum and minimum values of $y = x^4 - 5x^2 + 4$ on the interval $[1, 2]$.

SOLUTION. Since $y' = 4x^3 - 10x = 2x(2x^2 - 5)$ we see $y' = 0$ when $x = 0, \pm\sqrt{5/2}$. The only solution in the domain is $\sqrt{5/2}$. Note that $f(1) = 0$, $f(\sqrt{5/2}) = -9/4$, $f(2) = 0$, so the maximum is 0 and the minimum is $-9/4$.

8. True or false?

- (a) $\exp(\ln x) = x$ for every real number x .
False. When $x \leq 0$ the left-hand side is undefined.
- (b) $\ln(\exp(x)) = x$ for every real number x .
True.
- (c) If f is continuous, then f is differentiable.
False. Consider $f(x) = |x|$.
- (d) If f is differentiable, then f is continuous.
True.
- (e) If $f'(x) > 0$ for all x , then f has no relative maxima.
True.

9. Acceleration due to gravity on the moon is approximately 5.3 ft/sec^2 . How long does it take an object to hit the moon if it is dropped from a height of 10 feet?

SOLUTION. The stated conditions give rise to the initial value problem

$$s''(t) = -5.3, \quad s'(0) = 0, \quad s(0) = 10.$$

Its solution is given by the following:

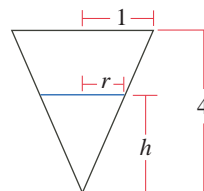
$$\begin{aligned} s'(t) &= -5.3t + C_1 \\ 0 &= s'(0) - C_1 \implies s'(t) = -5.3t \end{aligned}$$

$$\begin{aligned} s(t) &= \frac{-5.3}{2}t^2 + C_2 \\ 10 &= s(0) = C_2 \implies s(t) = \frac{-5.3}{2}t^2 + 10 \end{aligned}$$

To find out when the object hits the moon, solve $s(t) = 0$ to see $t = 2\sqrt{\frac{5}{5.3}}$ s.

10. A conical tank with its vertex pointing down has height 4 m and radius 1 m at the top. Oil flows in at the rate of $0.05 \text{ m}^3/\text{min}$. When the depth is 2 m, how fast is the level rising? (The volume of a right circular cone of base radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

SOLUTION. Let V be the volume of the oil in the tank h be its depth and r be the radius of its surface.



Know: $\frac{dV}{dt} = \frac{1}{20}$

$$\frac{1}{4} = \frac{r}{h} \implies r = \frac{h}{4}$$

$$V = \frac{\pi r^2 h}{3} \implies V = \frac{\pi h^3}{48}$$

Want: $\frac{dh}{dt}$ when $h = 2$.

$$\frac{dV}{dt} = \frac{\pi h^2}{16} \frac{dh}{dt}$$

When $h = 2$,

$$\frac{1}{20} = \frac{4\pi}{16} \frac{dh}{dt} \implies \frac{dh}{dt} = \frac{1}{5\pi} \text{ m/min}$$

11. Evaluate the following limits

(a) $\lim_{x \rightarrow 0} \frac{2x}{\tan 3x}$.

$$\lim_{x \rightarrow 0} \frac{2x}{\tan 3x} = \lim_{x \rightarrow 0} \frac{2}{3 \sec^2 3x} = \frac{2}{3}$$

(b) $\lim_{x \rightarrow \infty} e^x/x^4$

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{e^x}{x^4} &= \lim_{x \rightarrow \infty} \frac{e^x}{4x^3} \\ &= \lim_{x \rightarrow \infty} \frac{e^x}{12x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{24x} \\ &= \lim_{x \rightarrow \infty} \frac{e^x}{24} = \infty \end{aligned}$$

(c) $\lim_{x \rightarrow 0} \frac{x + \tan x}{\sin x}$

$$\lim_{x \rightarrow 0} \frac{x + \tan x}{\sin x} = \lim_{x \rightarrow 0} \frac{1 + \sec^2 x}{\cos x} = 2$$

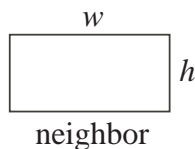
12. Express the following limit as a definite integral on the interval $[1, 7]$.

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(2x_i^* + 3\sqrt{x_i^*} \right) \frac{6}{n}$$

SOLUTION. $\int_1^7 (2x + 3\sqrt{x}) dx$

13. You wish to fence in a rectangular pasture of 30,000 square feet, with one side on a neighbor's lot. If the neighbor agrees to pay for half the cost of the fence that abuts his lot, what dimensions for the fence will minimize your cost?

SOLUTION. Suppose the dimensions of the rectangle are h and w , where the neighbor shares a side of length w . Let C be the total cost and fencing costs a dollars per foot.



Know: $hw = 30000 \implies h = 30000/w$

$$C = 2ah + \frac{3a}{2}w = \frac{60000a}{w} + \frac{3a}{2}w$$

Want: The minimum of C for $w > 0$.

$$C'(w) = -\frac{60000a}{w^2} + \frac{3a}{2}$$

$$C'(w) = 0 \implies w = 200$$

Since

$$C''(w) = \frac{120000a}{w^3} > 0, w > 0$$

it follows that $w = 200$ must be where a minimum occurs. This shows the field should be 150×200 with the long side facing the neighbor.

14. Use Newton's method to estimate the two zeros of the function $f(x) = x^4 + x - 3$. Start with $x_0 = -1$ for the zero on the left and $x_0 = 1$ for the zero on the right.

SOLUTION. The iterating function is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{3(x_n^4 + 1)}{4x_n^3 + 1}$$

Letting $x_0 = -1$ gives the following sequence:

$$-1., -2., -1.64516, -1.48572, -1.45381, \\ -1.45263, -1.45263, -1.45263, -1.45263$$

Letting $x_0 = 1$ gives the following sequence:

$$1., 1.2, 1.16542, 1.16404, 1.16404, 1.16404$$

Both of these are accurate to 5 decimal places.

15. Differentiate each of the following

(a) $y = x^7 + \sqrt{7}x - \frac{1}{\pi + 1}$

$$y' = 7x^6 + \sqrt{7}$$

(b) $y = 2 \tan^2 x - \sec^2 x$

$$y' = 2 \sec^2 x \tan x$$

(c) $y = (x^2 \cot 5x)^{1/3}$

$$y' = \frac{2x \cot(5x) - 5x^2 \csc^2(5x)}{3(x^2 \cot(5x))^{2/3}}$$

(d) $y = \log_2(x^2/2)$

$$y' = \frac{2}{x \ln 2}$$

(e) $y = t \tan^{-1} t - \frac{1}{2} \ln t$

$$y' = \frac{t}{t^2 + 1} + \tan^{-1}(t) - \frac{1}{2t}$$

16. Simplify $(\log_4 x)/(\log_2 x)$.

$$\frac{\log_4 x}{\log_2 x} = \frac{\ln x / \ln 4}{\ln x / \ln 2} = \frac{\ln x / (2 \ln 2)}{\ln x / \ln 2} = \frac{1}{2}$$

17. On the moon the acceleration of gravity is 1.6 m/sec^2 . If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 sec later?

SOLUTION. $1.6 \times 30 = 48 \text{ m/sec}^2$

18. Do the following antiderivatives.

$$(a) \int \left(8y - \frac{2}{y^{1/4}} \right) dy$$

$$\begin{aligned} \int \left(8y - \frac{2}{y^{1/4}} \right) dy &= \int \left(8y - 2y^{-1/4} \right) dy \\ &= 8 \frac{y^2}{2} - 2 \frac{y^{3/4}}{3/4} + C \\ &= 4y^2 - \frac{8}{3} y^{3/4} + C \end{aligned}$$

$$(b) \int x^{-3}(x+1) dx$$

$$\begin{aligned} \int x^{-3}(x+1) dx &= \int (x^{-2} + x^{-3}) dx \\ &= \frac{x^{-1}}{-1} + \frac{x^{-2}}{-2} + C \\ &= \frac{-1}{x} - \frac{1}{2x^2} + C \end{aligned}$$

$$(c) \int (-3 \csc^2 x) dx$$

$$\int (-3 \csc^2 x) dx = 3 \cot x + C$$

19. If $f''(x) = -1$, $f'(0) = -5$ and $f(0) = 2$, then find $f(x)$.

SOLUTION.

$$\begin{aligned} f'(x) &= \int (-1) dx = -x + C_1 \\ -5 &= f'(0) = C_1 \implies f'(x) = -x - 5 \\ f(x) &= \int (-x - 5) dx = -\frac{x^2}{2} - 5x + C_2 \\ 2 &= f(0) = C_2 \implies f(x) = -\frac{x^2}{2} - 5x + 2 \end{aligned}$$

20. Given the three functions $F_1(x) = \frac{x^2}{2} \sin x + C$, $F_2(x) = -x \cos x + C$ and $F_3(x) = -x \cos x + \sin x + C$, which is the antiderivative for $f(x) = x \sin x$? Explain your answer.

SOLUTION. The answer is F_3 because $F_3'(x) = f(x)$.

21. A particle moves on a number line with acceleration $a = 15\sqrt{t} - 3/\sqrt{t}$, subject to the conditions that $ds/dt = 4$ and $s = 0$ when $t = 1$. Find the position function s .

SOLUTION. The conditions of the problem give rise to the initial value problem

$$s''(t) = 15\sqrt{t} - 3/\sqrt{t}, \quad s'(1) = 4, \quad s(1) = 0.$$

Antidifferentiating and using the first initial condition gives

$$\begin{aligned} s'(t) &= \int s''(t) dt = 10t^{3/2} - 6t^{1/2} + C_1 \\ 4 &= s'(1) = 4 + C_1 \implies C_1 = 0 \\ s'(t) &= 10t^{3/2} - 6t^{1/2} \end{aligned}$$

Do the same with s' .

$$\begin{aligned} s(t) &= \int s'(t) dt = 4t^{5/2} - 4t^{3/2} + C_2 \\ 0 &= s(1) = C_2 \implies s(t) = 4t^{5/2} - 4t^{3/2} \end{aligned}$$

22. Find a value of c satisfying the conclusion of the Mean Value Theorem for the function $f(x) = \sqrt{x-1}$ on the interval $[1, 3]$.

SOLUTION.

$$\begin{aligned} f(3) - f(1) &= f'(c)(3-1) \\ \sqrt{2} - 0 &= \frac{2}{2\sqrt{c-1}} \\ c &= \frac{3}{2} \end{aligned}$$

23. Graph the function $y = x^4 - 4x^3 + 10$. Be sure to label critical points and inflection points.

SOLUTION. First, find the critical numbers.

$$\begin{aligned} y' &= 4x^3 - 12x^2 = 4x^2(x-3) \\ x &= 0, 3 \text{ are the critical numbers} \end{aligned}$$

Next, find the possible inflection points.

$$\begin{aligned} y'' &= 4x^3 - 12x^2 = 12x(x-1) \\ x &= 0, 1 \text{ are the possible inflection points} \end{aligned}$$

