Homework Problems

MATH 502, Spring 2009

Problem 1 (January 8)**.** Let G be an open set and $f \in D(G)$. If there is an $a \in G$ such that $\lim_{x\to a} f'(x)$ exists, then $\lim_{x\to a} f'(x) = f'(a)$.

Extra Credit 1 (January 8)**.** Prove or give a counter example: If $f \in D((a,b))$ such that f' is bounded, then there is an $F \in C([a,b])$ such that $f = F$ on (a,b).

Problem 2 (January 13)**.** Suppose f is continuous on $[a,b]$ and f'' exists on (a,b). If there is an $x_0 \in (a,b)$ such that the line segment between $(a, f(a))$ and $(b, f(b))$ contains the point $(x_0, f(x_0))$, then there is a $c \in (a,b)$ such that $f''(c) = 0$.

Problem 3 (January 13)**.** Prove that

$$\left|\sin x - \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\right| < \frac{1}{5040}$$

when $|x| \leq 1$.

Extra Credit 2 (January 15)**.** Prove or give a counter example: If f is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R} \setminus \{0\}$ with $\lim_{x\to 0} f'(x) = L$, then f is differentiable on $\mathbb{R}$.

Problem 4 (January 15)**.** Let f be defined on a neighborhood of x.

(a) If $f''(x)$ exists, then

$$\lim_{h\to 0} \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} = f''(x).$$

(b) Find a function f where this limit exists, but $f''(x)$ does not exist.

Problem 5 (January 20)**.** If $f : D \to \mathbb{R}$ is uniformly continuous on a bounded set D, then f is bounded.

Problem 6 (January 22)**.** A Riemann integrable function is bounded.

Problem 7 (February 3)**.** Calculate $\int_0^1 x^2$.

Extra Credit 3 (February 3)**.** In some calculus books, the definition of the integral is given as follows.

> Let f be a function on $[a,b]$, $n \in \mathbb{N}$, P_n be the regular partition of $[a,b]$ with n subintervals and m_k be the midpoint of interval I_k determined by P. If
>
> $$\lim_{n\to\infty} \sum_{k=1}^{n} f(m_k)\frac{b-a}{n} = L,$$
>
> then f is integrable on $[a,b]$ and $\int_a^b f(x)\,dx = L$.

Is this equivalent to our definition? Is it a practical definition?

Problem 8 (February 5)**.** If f is integrable on $[a,b]$ and $\int fg = 0$ for all continuous functions g, then $f(x) = 0$ for all $x \in C(f)$.

Problem 9 (February 10)**.** If $f, g : [a,b] \to \mathbb{R}$, $\int_a^b f$ exists and $\{x \in [a,b] : f(x) \neq g(x)\}$ is finite, then $\int_a^b g$ exists and $\int_a^b f = \int_a^b g$.

Problem 10 (February 12)**.** For $x > 0$, define $\ln x = \int_1^x \frac{1}{t}$. Prove that when $a > 0$ and $b > 0$, then $\ln(ab) = \ln a + \ln b$.

Problem 11 (February 17)**.** If $f, g : [a,b] \to \mathbb{R}$ are both integrable, then $f(x) \vee g(x)$ and $f(x) \wedge g(x)$ are both integrable.

Extra Credit 4 (February 17)**.** If

$$f(x) = \begin{cases} 0, & x \le 0 \\ \sin\left(\frac{\pi}{x}\right), & x > 0 \end{cases} \text{ and } g(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases},$$

then f has an antiderivative and g has no antiderivative.

Problem 12 (February 19)**.** If $f : [a,b] \to \mathbb{R}$ is monotone, then $\int_a^b f$ exists.

Problem 13 (February 24)**.** If $c(x)$ is the Cantor function, then what is $\int_0^1 c$?

Problem 14 (February 26)**.** If ℓ_n is a sequence of linear functions converging pointwise to a function f on $\mathbb{R}$, then f is linear.

Problem 15 (March 3)**.** $\sum_{n=0}^{\infty} x^n$ converges uniformly on every compact subset of its domain.

Problem 16 (March 3)**.** A sequence of functions $f : S \to \mathbb{R}$ converges *locally uniformly*, if for each $x \in S$, there is a neighborhood G_x of x such that f_n converges uniformly on G_x. Show that if $f_n : S \to \mathbb{R}$ is a sequence of continuous functions converging locally uniformly to a function f, then f is continuous. Find a sequence f_n that converges locally uniformly, but not uniformly.

Problem 17 (March 5)**.** Where is $\sum_{n=0}^{\infty} e^{-nx} \cos nx$ continuous?

Problem 18 (March 10)**.** If $f \in C([0,1])$, then evaluate

$$\lim_{n\to\infty} \int_0^1 x^n f(x)\, dx.$$

Extra Credit 5 (March 10)**.** If $f \in C([0,1])$, then evaluate

$$\lim_{n\to\infty} n \int_0^1 x^n f(x)\, dx.$$

Extra Credit 6 (March 24)**.** Prove or give a counter example: If $r > 0$ and the series $\sum_{n=1}^{\infty} a_n x^n$ converges everywhere on $[0, r]$, then it converges uniformly on $[0, r]$.

Problem 19 (March 24)**.** Find an example of a function $f(x) = \sum_{n=1}^{\infty} a_n x^n$ such that the power series for f and f' have different domains.

Problem 20 (March 26)**.** What is the Abel sum of $1 - 2 + 3 - 4 + \cdots$?

Problem 21 (April 2)**.** Let $\sum_{n=1}^{\infty} a_n$ be a series with partial sums s_n. Define the sequence

$$\sigma_n = \frac{1}{n} \sum_{k=1}^{n} s_k.$$

Prove that if $s_n \to s$, then $\sigma_n \to s$. Give an example to show the converse is not true.[1]

Problem 22 (April 7)**.** Can the three functions x, x^2 and x^3 be elements of a set of functions orthogonal on some interval $[a, b]$?

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[1] This method of summing a series is known as Cesàro summability. When a series has a limit by this method, it is said to summable $(C, 1)$. If $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sigma_n$ exists, then the series is summable $(C, 2)$. This process can be continued to define summability (C, n) for any $n \in \mathbb{N}$.

Problem 23 (April 7)**.** Find four polynomials orthonormal on $[-1, 1]$.

Problem 24 (April 9)**.** If f is integrable on $[-\pi, \pi]$ and the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

converges uniformly to f, then it is the Fourier series for f.