MATH 502, Spring 2009 Homework Solutions

Problem 1 (January 8)**.** Let G be an open set and $f \in D(G)$. If there is an $a \in G$ such that $\lim_{x \to a} f'(x)$ exists, then $\lim_{x \to a} f'(x) = f'(a)$.

Here are two solutions.

Proof 1. Let b_n be a sequence contained in $G \setminus \{a\}$ such that $b_n \to a$. According to the Mean Value Theorem, for each $n \in \mathbb{N}$, there is a $c_n \in (a \wedge b_n, a \vee b_n)$ such that $f(b_n) - f(a) = f'(c_n)(b_n - a)$. The Squeeze Theorem guarantees that $c_n \to a$ and $f'(c_n) \to \lim_{x \to a} f'(x)$ by assumption. Finally,

$$f'(a) = \lim_{n \to \infty} \frac{f(b_n) - f(a)}{b_n - a} = \lim_{n \to \infty} f'(c_n) = \lim_{x \to a} f'(x).$$

□

Proof 2. If $\lim_{x \to a} f'(x) \neq f'(a)$, then f' has a jump discontinuity at a, which violates Theorem 7.3.6. □

Extra Credit 1 (January 8)**.** Prove or give a counter example: If $f \in D((a,b))$ such that f' is bounded, then there is an $F \in C([a,b])$ such that $f = F$ on (a,b).

Proof. In order to find a possible value for $f(a)$, suppose $B > 0$ such that $|f'(x) \leq B$ when $x \in (a,b)$ and let $\varepsilon > 0$. If a_n is a sequence from (a,b) such that $a_n \to a$, then there must exist an $N \in \mathbb{N}$ such that $n, m \geq N \implies |a_n - a_m| < \varepsilon / B$. Using this and the Mean Value Theorem, we see that when $n, m \geq N$, then there is a $c \in (a,b)$ such that

$$|f(a_n) - f(a_m)| = |f'(c)(a_n - a_m)| < B \frac{\varepsilon}{B} = \varepsilon.$$

So, $f(a_n)$ is a Cauchy sequence and must converge to some L.

Let $F(a) = L$ and $F(x) = f(x)$ for $x \in (a,b)$. It must be shown that $a \in C(F)$.

To this end, let x_n and ε be as above and y_n another sequence from (a,b) with $y_n \to a$. From the Mean Value Theorem, for each $n \in \mathbb{N}$ there is a $c_n \in (a,b)$ such that

$$|f(x_n) - f(y_n)| = |f'(c_n)(x_n - y_n)| \leq B|x_n - y_n| \to B|a - a| = 0.$$

This implies $f(y_n) \to L = F(a)$. An application of Corollary 6.3.3 shows $a \in C(F)$.

A similar argument establishes a value for $F(b)$ making $F \in C([a,b])$. □

Problem 2 (January 13)**.** Suppose f is continuous on $[a,b]$ and f'' exists on (a,b). If there is an $x_0 \in (a,b)$ such that the line segment between $(a, f(a))$ and $(b, f(b))$ contains the point $(x_0, f(x_0))$, then there is a $c \in (a,b)$ such that $f''(c) = 0$.

Proof. Let $\ell(x)$ be the line determined by $(a, f(a))$ and $(b, f(b))$ and $F(x) = f(x) - \ell(x)$. Then $F \in C([a,b] \cap D((a,b))$ and $F(a) = F(x_0) = F(b) = 0$. According to Rolle's Theorem there are $c_1 \in (a, x_0)$ and $c_2 \in (x_0, b)$ such that $F'(c_1) = F'(c_2) = 0$. Since $f' \in D((a,b))$, it must be that $F' \in D((a,b))$ and $F' \in C([c_1, c_2])$. Another application of Rolle's Theorem gives a $c \in (c_1, c_2)$ such that

$$0 = F''(c) = f''(c) + \ell''(c) = f''(c),$$

because ℓ is a linear function. □

Problem 3 (January 13)**.** Prove that

$$\left|\sin x - \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\right| < \frac{1}{5040}$$

when $|x| \leq 1$.

Proof. If $f(x) = \sin x$, Taylor's Theorem centered at $a = 0$ with $n = 6$ says there is a c between 0 and x such that

$$\left|\sin x - \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\right| = \left|\frac{\sin c}{7!} x^7\right| < \frac{1}{7!} = \frac{1}{5040}.$$

□

Extra Credit 2 (January 15)**.** Prove or give a counter example: If f is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R} \setminus \{0\}$ with $\lim_{x\to 0} f'(x) = L$, then f is differentiable on $\mathbb{R}$.

Proof. If $b > a$ then both $G(x) = x - a$ and $F(x) = f(x) - f(a)$ are continuous on $[a,b]$ and differentiable on (a,b) with $\lim_{x\downarrow a} G(x) = \lim_{x\downarrow a} F(x) = 0$. Since, by assumption,

$$L = \lim_{x\downarrow a} f'(x) = x \downarrow a \frac{F'(x)}{G'(x)},$$

it follows from L'Hôspital's Rule, Theorem 7.4.2, that

$$x \downarrow a \frac{f(x) - f(a)}{x - a} = x \downarrow a \frac{F(x)}{G(x)} = L. \tag{1.1}$$

A similar argument establishes

$$x \uparrow a \frac{f(x) - f(a)}{x - a} = x \downarrow a \frac{F(x)}{G(x)} = L. \tag{1.2}$$

In light of (1.1) and (1.2), it follows that $f'(a) = L$. □

Problem 4 (January 15)**.** Let f be defined on a neighborhood of x.

(a) If $f''(x)$ exists, then

$$\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x). \tag{1.3}$$

(b) Find a function f where this limit exists, but $f''(x)$ does not exist.

Solution. (a) Since $f''(x)$ exists,

$$\begin{aligned} f''(x) &= \lim_{h \to 0} \frac{1}{2} \left(\frac{f'(x+h) - f'(x)}{h} + \frac{f'(x) - f'(x-h)}{h} \right) \\ &= \lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h}. \end{aligned} \tag{1.4}$$

Since f' must exist on a neighborhood of x, both h^2 and $f(x+h) - 2f(x) + f(x-h)$ are continuous and differentiable, as functions of h, on a neighborhood of $h = 0$. An application of L'Hôspital's Rule, Theorem 7.4.2, combined with (1.4) shows

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h} = \lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

(b) An example is

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

when $x = 0$. Any function odd about x such that $f''(x)$ fails to exist is an example. □

Problem 5 (January 20)**.** If $f : D \to \mathbb{R}$ is uniformly continuous on a bounded set D, then f is bounded.

Proof. Suppose not. Then, for each $n \in \mathbb{N}$ there is an $x_n \in D$ such that $|f(x_n)| > n$. The Balzano-Weierstrass Theorem gives a convergent subsequence x_{n_k}. Since $|f(x_{n_k}| > n_k \to \infty$, this contradicts Theorem 6.6.4. □

Problem 6 (January 22)**.** A Riemann integrable function is bounded.

Proof. Suppose $f : [a,b] \to \mathbb{R}$ is unbounded above, $B > 0$ and $\delta > 0$. For each $n \in \mathbb{N}$ there is an $x_n \in [a,b]$ so that $f(x_n) > n$. The Bolzano Weierstrass theorem gives a convergent subsequence of x_n. We may as well assume x_n is already this subsequence and $x_n \to x_0$. Let I be an open interval with $x_0 \in I$ and $|I| < \delta$. Choose a partition P of $[a,b]$, with $\|P\| < \delta$ and $\overline{I} \cap [a,b] = I_{k_0}$, where I_{k_0} is one of the intervals determined by P.

Assuming P is the generic partition, select $\alpha_k \in I_k$ for $k \neq k_0$. If $\sum_{k \neq k_0} f(\alpha_k)|I_k| = c$, then choose $\alpha_{k_0} = x_n$ so $x_n \in I_{k_0}$ and $n|I_{k_0}| > B - c$. With this selection,

$$\mathcal{R}(f, P, \alpha_k) = f(x_n)|I_{k_0}| + \sum_{k \neq k_0} f(\alpha_k)|I_k| > B - c + c = B.$$

Therefore, for every $B > 0$ and $\delta > 0$, there is always a partition P with $\|P\| < \delta$ and $\mathcal{R}(f, P, \alpha_k) > B$ for some selection α_k from P. It follows that f is not Riemann integrable. □

Problem 7 (February 3)**.** Calculate $\int_0^1 x^2$.

Proof. According to Corollary 8.3.3 and Theorem 8.4.1, the integral $\int_0^1 x^2$ exists. Let $n \in \mathbb{N}$ and $P = \{i/n : 0 \leq i \leq n\}$ be the regular partition of $[0,1]$ into n subintervals. Then $m_i = f(\frac{i-1}{n}) = \frac{(i-1)^2}{n^2} < \frac{i^2}{n^2} f(\frac{i}{n} = M_n$,

$$\underline{\mathcal{D}}(f, P) = \sum_{i=1}^{n} m_n |I_i| = \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{(n-1)(n)(2(n-1)+1)}{6n^3} = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$$

converges to $1/3$ as $n \to \infty$. □

Extra Credit 3. In some calculus books, the definition of the integral is given as follows.

> Let f be a function on $[a,b]$, $n \in \mathbb{N}$, P_n be the regular partition of $[a,b]$ with n subintervals and m_k be the midpoint of interval I_k determined by P. If
>
> $$\lim_{n \to \infty} \sum_{k=1}^{n} f(m_k) \frac{b-a}{n} = L,$$
>
> then f is integrable on $[a,b]$ and $\int_a^b f(x)\,dx = L$.

Is this equivalent to our definition? Is it a practical definition?

Solution. Let $f(x) = \mathcal{X}_{\mathbb{Q}}$ and suppose $a, b \in \mathbb{Q}$ so that $m_k \in \mathbb{Q}$ for all k. Then

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(m_k) \frac{b-a}{n} = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{b-a}{n} = 1,$$

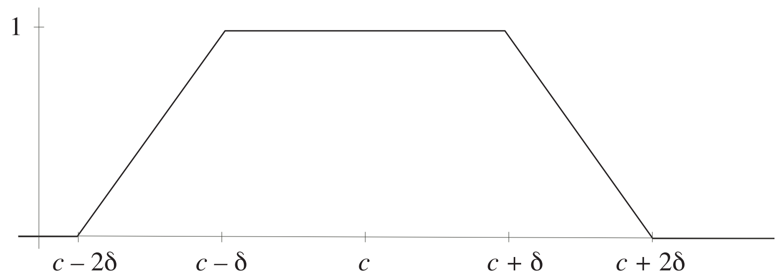
but f is not Riemann integrable because $\overline{\mathcal{D}}(f) = 1 > 0 = \underline{\mathcal{D}}(f)$.

It is practical for a beginning calculus course where all the functions encountered are Riemann integrable because both definitions give the same answers for such functions. □

Problem 8 (February 5)**.** If f is integrable on $[a,b]$ and $\int fg = 0$ for all continuous functions g, then $f(x) = 0$ for all $x \in C(f)$.

Proof. Suppose $f(c) > 0$ for some $c \in (a,b) \cap C(f)$. There is a $\delta > 0$ such that $(c - 2\delta, c + 2\delta) \subset (a,b)$ and $x \in (c - 2\delta, c + 2\delta) \implies f(x) > f(c)/2$. Define

$$g(x) = \begin{cases} 0, & |x - c| \ge 2\delta \\ 1, & |x - c| \le \delta \\ \frac{x-(c-2\delta)}{\delta}, & x \in (c - 2\delta, c - \delta) \\ -\frac{x-(c+2\delta)}{\delta}, & x \in (c + \delta, c + 2\delta) \end{cases}.$$



If $P \in \mathcal{P}([a,b])$ with $\{c-\delta, c+\delta\} \subset P$, then there are natural numbers $j < k$ so $c - \delta = x_j < x_k = c + \delta$. Note that

$$\mathcal{R}(fg, P, x_i^*) \ge \sum_{i=j}^{k} f(x_i^*)g(x_i^*) > \delta f(c)$$

for every such partition. This shows $\int_a^b fg \ge \delta f(c) > 0$, which is a contradiction. So, $f(c) = 0$ whenever $c \in C(f)$. □

Problem 9 (February 10)**.** If $f, g : [a,b] \to \mathbb{R}$, $\int_a^b f$ exists and $\{x \in [a,b] : f(x) \ne g(x)\}$ is finite, then $\int_a^b g$ exists and $\int_a^b f = \int_a^b g$.

Proof. Let $\varepsilon > 0$, $h = f - g$ and $U = \{x : h(x) \ne 0\}$. By assumption, $\operatorname{card}(U) \in \omega$. If $\operatorname{card}(U) = 0$, there's nothing to prove, so assume $\operatorname{card}(U) = \ell \in \mathbb{N}$. Since U is finite, $M = \max\{|h(x)| : x \in U\}$ is well-defined. Choose $\delta \in (0, \varepsilon/2M\ell)$ and $P \in \mathcal{P}([a,b])$ with $\|P\| < \delta$. Then, using the standard notation with the generic partition,

$$\begin{aligned}\overline{\mathcal{D}}(h,P) = \sum_{i=1}^{n} M_i|I_i| &= \sum_{U \cap I_i = \emptyset} M_i|I_i| + \sum_{U \cap I_i \ne \emptyset} M_i|I_i| \\ &= \sum_{U \cap I_i \ne \emptyset} M_i|I_i| \le M \sum_{U \cap I_i \ne \emptyset} |I_i| < M\ell\delta < \frac{\varepsilon}{2}\end{aligned}$$

In the same way, it is shown that $\underline{\mathcal{D}}(h,P) > -\varepsilon/2$. Since $\overline{\mathcal{D}}(h,P) - \underline{\mathcal{D}}(h,P) < \varepsilon$, Corollary 8.3.2 implies $\int_a^b h$ exists and $-\varepsilon/2 < \underline{\mathcal{D}}(h,P) \le \overline{\mathcal{D}}(h,P) < \varepsilon/2$ implies $\int_a^b h = 0$. Finally, Theorem 8.6.1 yields

$$\int_a^b f = \int_a^b f - \int_a^b h = \int_a^b (f-h) = \int_a^b g.$$

□

Problem 10 (February 12)**.** For $x > 0$, define $\ln x = \int_1^x \frac{1}{t}$. Prove that when $a > 0$ and $b > 0$, then $\ln(ab) = \ln a + \ln b$.

Proof. According to the Chain Rule and the Fundamental Theorem of Calculus, $\frac{d}{dx}\ln(ax) = \frac{1}{x} = \frac{d}{dx}(\ln(a)+\ln(x))$. According to Corollary 7.3.5, there is a $c \in \mathbb{R}$ such that $\ln(ax) = \ln(a) + \ln(x) + c$ for all $x > 0$. Substituting $x = 1$ shows $c = 0$. □

Problem 11 (February 17)**.** If $f, g : [a,b] \to \mathbb{R}$ are both integrable, then $f(x) \vee g(x)$ and $f(x) \wedge g(x)$ are both integrable.

Proof. Noting that $f \vee g = \frac{1}{2}(f + g + |f - g|)$ and $f \wedge g = \frac{1}{2}(f + g - |f - g|)$, this follows from parts (a) and (c) of Theorem 8.6.1. □

Extra Credit 4 (February 17)**.** If

$$f(x) = \begin{cases} 0, & x \le 0 \\ \sin\left(\frac{\pi}{x}\right), & x > 0 \end{cases} \text{ and } g(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases},$$

then f has an antiderivative and g has no antiderivative.

Proof. It is clear from Darboux's Theorem that g cannot be a derivative.

Let $F(x) = \int_{-1}^x f$. The Fundamental Theorem of Calculus shows $F'(x) = f(x)$ whenever $x \ne 0$ and it is clear that $\lim_{h\uparrow 0} \frac{F(0+h)-F(0)}{h} = 0$. In order to conclude $F'(0) = 0$, it suffices to show that $\lim_{h\downarrow 0} \frac{F(0+h)-F(0)}{h} = 0$.

To do this, let $a_n = \int_{1/(n+1)}^{1/n} f$. Evidently, a_n alternates in sign and $|a_n| \downarrow 0$, so $F(1/n) = \sum_{i=n}^\infty a_i$ is a convergent alternating series. Note that as a consequence of the Alternating Series Test,

$$|a_n| > |\sum_{i=n+1}^\infty a_i|, \ \forall n \in \mathbb{N}.$$

If $\frac{1}{n+1} \leq h < \frac{1}{n}$, then

$$\begin{aligned}\left|\frac{1}{h}\int_0^h f\right| &= \frac{1}{h}\left|\sum_{i=n+1}^{\infty} a_i + \int_{1/(n+1)}^{h} f\right| \\ &\leq \frac{1}{h}\left(|a_n| + \int_{1/(n+1)}^{n} |f|\right) \\ &\leq \frac{2|a_n|}{h} \leq \frac{1}{n+1}\left(\frac{1}{n} - \frac{1}{n+1}\right) \to 0.\end{aligned}$$

□

Problem 12 (February 19)**.** If $f : [a, b] \to \mathbb{R}$ is monotone, then $\int_a^b f$ exists.

Proof. If $f(a) = f(b)$, then f is constant and we're done. So, assume $f(a) < f(b)$. Let $\varepsilon > 0$ and P be a regular partition of $[a, b]$ with $\|P\| < \varepsilon/(f(b)-f(a))$. Then, thinking of P as the generic partition,

$$\begin{aligned}\overline{\mathcal{D}}(f, P) - \underline{\mathcal{D}}(f, P) &= \sum_{k=1}^{n}(M_k - m_k)|I_k| \\ &= \|P\|\sum_{k=1}^{n}(f(x_k) - f(x_{k-1} \\ &= \|P\|(f(b) - f(a)) < \varepsilon\end{aligned}$$

Now use Corollary 8.3.2 to finish the proof.

Of course, this is also an immediate consequence of Lebesgue's characterization of the Riemann integral and Theorem 6.4.2. □

Problem 13 (February 24)**.** If $c(x)$ is the Cantor function, then what is $\int_0^1 c$?

Solution. Let $I_1^1 = [1/3, 2/3]$, $I_2^1 = [1/9, 2/9]$, $I_2^2 = [7/9, 8/9]$, and, in general, I_n^k be the 2^{n-1} intervals each of length $1/3^n$ complementary to the Cantor set in $[0, 1]$ and arranged in order by increasing endpoints. The Cantor function has value $(2k - 1)/2^n$ on I_n^k, so

$$\sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}}\frac{2k-1}{2^n}\frac{1}{3^n} = \sum_{n=1}^{\infty}\frac{1}{6^n}\sum_{k=1}^{2^{n-1}}(2k-1) = \sum_{n=1}^{\infty}\frac{2^n}{6^n} = \frac{1}{2}$$

□

Problem 14 (February 26)**.** If ℓ_n is a sequence of linear functions converging pointwise to a function f on $\mathbb{R}$, then f is linear.

Proof. Let $\ell_n(x) = m_n x + b_n$. When $x = 0$, then $\ell_n(0) = b_n \to f(0)$. Using this, we see

$$\ell(1) - b_n = m_n \to f(1) - f(0).$$

Let the linear function $\ell(x) = (f(1) - f(0)x + f(0)$ and $a \in \mathbb{R}$. Then

$$\ell_n(a) = m_n a + b_n \to (f(1) - f(0))a + f(0) = f(a)$$

shows $f = \ell$. □

Problem 15 (March 3)**.** $\sum_{n=0}^{\infty} x^n$ converges uniformly on every compact subset of its domain.

Proof. According to the ratio test, the domain of $\sum_{n=0}^{\infty} x^n$ is $(-1, 1)$. If K is a compact subset of $(-1, 1)$, then there is a $B \in (0, 1)$ such that $K \subset [-B, B] \subset (-1, 1)$. If $x \in K$, then $|x|^n \leq B^n$. Since $\sum_{n=0}^{\infty} B^n$ converges, we see $\sum_{n=0}^{\infty} x^n$ converges uniformly on $[-B, B]$ by the Weierstrass M-test. It will obviously converge uniformly on every subset of $[-B, B]$. □

Problem 16 (March 3)**.** A sequence of functions $f : S \to \mathbb{R}$ converges *locally uniformly*, if for each $x \in S$, there is a neighborhood G_x of x such that f_n converges uniformly on G_x. Show that if $f_n : S \to \mathbb{R}$ is a sequence of continuous functions converging locally uniformly to a function f, then f is continuous. Find a sequence f_n that converges locally uniformly, but not uniformly.

Proof. Let $x \in S$ and U be a neighborhood of x on which f_n converges uniformly. Theorem 9.5.1 shows $x \in C(f)$.

Example 9.1.3 converges locally uniformly, but not uniformly. □

Problem 17 (March 5)**.** Where is $\sum_{n=0}^{\infty} e^{-nx} \cos nx$ continuous?

Solution. Let $f(x) = \sum_{n=0}^{\infty} e^{-nx} \cos nx$. The domain of the series is clearly $(0, \infty)$. We claim it is continuous on its domain. To see this, let $c > 0$. For $x \geq c$,

$$\left|e^{-nx} \cos nx\right| \leq e^{-nx} \leq e^{-nc}.$$

Because $\sum_{n=0}^{\infty} e^{-nc} = 1/(1-e^{-c})$, the Weierstrass M-test, Theorem 9.4.3, shows f is uniformly convergent on $[c, \infty)$. Corollary 9.5.2 implies f is continuous on (c, ∞). Since $c > 0$ is arbitrary, f must be continuous on $(0, \infty)$. □

Problem 18 (March 10)**.** If $f \in C([0, 1])$, then evaluate

$$\lim_{n\to\infty} \int_0^1 x^n f(x)\, dx.$$

Solution. Since f is continuous on the compact set $[0, 1]$, $\|f\|_{[0,1]} = M \geq 0$. Therefore,

$$\lim_{n\to\infty} \left| \int_0^1 x^n f(x)\, dx \right| \leq \lim_{n\to\infty} M \int_0^1 x^n\, dx = \lim_{n\to\infty} \frac{M}{n+1} = 0.$$

□

Extra Credit 5 (March 10)**.** If $f \in C([0,1])$, then evaluate

$$\lim_{n\to\infty} n \int_0^1 x^n f(x)\, dx.$$

Solution. Let $\varepsilon > 0$ and $\|f\|_{[0,1]} = M \geq 0$. Since f is continuous, there is a $\delta \in [0,1]$ so $|f(1) - f(x)| < \varepsilon$ whenever $x \in (\delta, 1)$. First, notice

$$\lim_{n\to\infty} \left| n \int_0^\delta x^n f(x)\, dx \right| \leq M \lim_{n\to\infty} n \int_0^\delta x^n\, dx = M \lim_{n\to\infty} \frac{n\,\delta^{n+1}}{n+1} = 0 \tag{1.5}$$

Second,

$$(f(1) - \varepsilon) n \int_\delta^1 x^n\, dx < n \int_\delta^1 x^n f(x)\, dx < (f(1) + \varepsilon) n \int_\delta^1 x^n\, dx$$

implies

$$(f(1) - \varepsilon) \frac{n(1 - \delta^{n+1})}{n+1} < n \int_\delta^1 x^n f(x)\, dx < (f(1) + \varepsilon) \frac{n(1 - \delta^{n+1})}{n+1} \tag{1.6}$$

Letting $n \to \infty$ in (1.6) gives

$$f(1) - \varepsilon < \lim_{n\to\infty} n \int_\delta^1 x^n f(x)\, dx < f(1) + \varepsilon \tag{1.7}$$

A combination of (1.5) and (1.7) shows

$$f(1) - \varepsilon < \lim_{n\to\infty} n \int_0^1 x^n f(x)\, dx < f(1) + \varepsilon.$$

Clearly, $\lim_{n\to\infty} n \int_0^1 x^n f(x)\, dx = f(1)$. □

Extra Credit 6 (March 24)**.** Prove or give a counter example: If $r > 0$ and the series $\sum_{n=1}^\infty a_n x^n$ converges everywhere on $[0, r]$, then it converges uniformly on $[0, r]$.

Problem 19 (March 24)**.** Find an example of a function $f(x) = \sum_{n=1}^\infty a_n x^n$ such that the power series for f and f' have different domains.

Problem 20 (March 26)**.** What is the Abel sum of $1 - 2 + 3 - 4 + \cdots$?

Problem 21 (April 2)**.** Let $\sum_{n=1}^\infty a_n$ be a series with partial sums s_n. Define the sequence

$$\sigma_n = \frac{1}{n} \sum_{k=1}^n s_k.$$

Prove that if $s_n \to s$, then $\sigma_n \to s$. Give an example to show the converse is not true.[2]

[2]This method of summing a series is known as Cesàro summability. When a series has a limit by this method, it is said to summable $(C, 1)$. If $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \sigma_n$ exists, then the series is summable $(C, 2)$. This process can be continued to define summability (C, n) for any $n \in \mathbb{N}$.