CHAPTER 5

The Topology of \( \mathbb{R} \)

1. Open and Closed Sets

**Definition 5.1.** A set \( G \subset \mathbb{R} \) is open if for every \( x \in G \) there is an \( \epsilon > 0 \) such that \( (x - \epsilon, x + \epsilon) \subset G \). A set \( F \subset \mathbb{R} \) is closed if \( F^c \) is open.

The idea is that about every point of an open set, there is some room inside the set on both sides of the point. It is easy to see that any open interval \((a, b)\) is an open set because if \( a < x < b \) and \( \epsilon = \min\{|x-a|, |x-b|\} \), then \( (x-\epsilon, x+\epsilon) \subset (a, b) \).

It’s obvious \( \mathbb{R} \) itself is an open set.

On the other hand, any closed interval \([a, b]\) is a closed set. To see this, it must be shown its complement is open. Let \( x \in [a, b]^c \) and \( \epsilon = \min\{|x-a|, |x-b|\} \). Then \( (x-\epsilon, x+\epsilon) \subset [a, b] \). Therefore, \( [a, b]^c \) is open, and its complement, namely \([a, b]\), is closed.

A singleton set \( \{a\} \) is closed. To see this, suppose \( x \neq a \) and \( \epsilon = |x-a| \). Then \( a \notin (x-\epsilon, x+\epsilon) \), and \( \{a\}^c \) must be open. The definition of a closed set implies \( \{a\} \) is closed.

Open and closed sets can get much more complicated than the intervals examined above. For example, similar arguments show \( \mathbb{Z} \) is a closed set and \( \mathbb{Z}^c \) is open. Both have an infinite number of disjoint pieces.

A common mistake is to assume all sets are either open or closed. Most sets are neither open nor closed. For example, if \( S = [a, b) \) for some numbers \( a < b \), then no matter the size of \( \epsilon > 0 \), neither \( (a-\epsilon, a+\epsilon) \) nor \( (b-\epsilon, b+\epsilon) \) are contained in \( S \) or \( S^c \).

**Theorem 5.2.** (a) Both \( \emptyset \) and \( \mathbb{R} \) are open.
(b) If \( \{G_\lambda : \lambda \in \Lambda\} \) is a collection of open sets, then \( \bigcup_{\lambda \in \Lambda} G_\lambda \) is open.
(c) If \( \{G_k : 1 \leq k \leq n\} \) is a finite collection of open sets, then \( \bigcap_{k=1}^n G_k \) is open.

**Proof.**
(a) \( \emptyset \) is open vacuously. \( \mathbb{R} \) is obviously open.
(b) If \( x \in \bigcup_{\lambda \in \Lambda} G_\lambda \), then there is a \( \lambda_x \in \Lambda \) such that \( x \in G_{\lambda_x} \). Since \( G_{\lambda_x} \) is open, there is an \( \epsilon > 0 \) such that \( x \in (x-\epsilon, x+\epsilon) \subset G_{\lambda_x} \subset \bigcup_{\lambda \in \Lambda} G_\lambda \). This shows \( \bigcup_{\lambda \in \Lambda} G_\lambda \) is open.
(c) If \( x \in \bigcap_{k=1}^n G_k \), then \( x \in G_k \) for \( 1 \leq k \leq n \). For each \( G_k \) there is an \( \epsilon_k \) such that \( (x-\epsilon_k, x+\epsilon_k) \subset G_k \). Let \( \epsilon = \min\{\epsilon_k : 1 \leq k \leq n\} \). Then \( (x-\epsilon, x+\epsilon) \subset G_k \) for \( 1 \leq k \leq n \), so \( (x-\epsilon, x+\epsilon) \subset \bigcap_{k=1}^n G_k \). Therefore \( \bigcap_{k=1}^n G_k \) is open.

Applying DeMorgan’s laws to the parts of Theorem 5.2 immediately yields the following.
COROLLARY 5.3. \(\text{(a)}\) Both \(\emptyset\) and \(\mathbb{R}\) are closed. \(\text{(b)}\) If \(\{F_{\lambda} : \lambda \in \Lambda\}\) is a collection of closed sets, then \(\bigcap_{\lambda \in \Lambda} G_{\lambda}\) is closed. \(\text{(c)}\) If \(\{F_k : 1 \leq k \leq n\}\) is a finite collection of closed sets, then \(\bigcup_{k=1}^n F_k\) is closed.

Surprisingly, \(\emptyset\) and \(\mathbb{R}\) are both open and closed. They are the only subsets of \(\mathbb{R}\) with this dual personality. Sets that are both open and closed are said to be **clopen**.

1.1. **Topological Spaces.** The preceding theorem provides the starting point for a fundamental area of mathematics called **topology.** The properties of the open sets of \(\mathbb{R}\) motivated the following definition.

**Definition 5.4.** For \(X\) a set, not necessarily a subset of \(\mathbb{R}\), let \(\mathcal{T} \subset \mathcal{P}(X)\). The set \(\mathcal{T}\) is called a **topology** on \(X\) if it satisfies the following three conditions.

\(\text{(a)}\) The union of any collection of sets from \(\mathcal{T}\) is also in \(\mathcal{T}\).
\(\text{(b)}\) The intersection of any finite collection of sets from \(\mathcal{T}\) is also in \(\mathcal{T}\).
\(\text{(c)}\) \(X \in \mathcal{T}\) and \(\emptyset \in \mathcal{T}\).

The pair \((X, \mathcal{T})\) is called a topological space. The elements of \(\mathcal{T}\) are the **open** sets of the topological space. The **closed sets** of the topological space are those sets whose complements are open.

If \(\mathcal{O} = \{G \subset \mathbb{R} : G\) is open\}\), then Theorem 5.2 shows \((\mathbb{R}, \mathcal{O})\) is a topological space and \(\mathcal{O}\) is called the **standard topology** on \(\mathbb{R}\). While the standard topology is the most widely used topology, there are many other possible topologies on \(\mathbb{R}\). For example, \(\mathcal{R} = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}\) is a topology on \(\mathbb{R}\) called the **right ray** topology. The collection \(\mathcal{F} = \{S \subset \mathbb{R} : S^c\) is finite\}\) \(\cup \{\emptyset\}\) is called the **finite complement** topology. The study of topologies is a huge subject, further discussion of which would take us too far afield. There are many fine books on the subject ([16]) to which one can refer.

1.2. **Limit Points and Closure.**

**Definition 5.5.** \(x_0\) is a **limit point**\(^1\) of \(S \subset \mathbb{R}\) if for every \(\varepsilon > 0\),

\((x_0 - \varepsilon, x_0 + \varepsilon) \cap S \setminus \{x_0\} \neq \emptyset.\)

The **derived set** of \(S\) is

\(S' = \{x : x\) is a limit point of \(S\}.\)

A point \(x_0 \in S \setminus S'\) is an **isolated point** of \(S.\)

Notice that limit points of \(S\) need not be elements of \(S\), but isolated points of \(S\) must be elements of \(S\). In a sense, limit points and isolated points are at

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\(^1\)This use of the term **limit point** is not universal. Some authors use the term **accumulation point.** Others use **condensation point,** although this is more often used for those cases when every neighborhood of \(x_0\) intersects \(S\) in an uncountable set.
The observations given above can be restated in terms of neighborhoods.

X0 is a limit point of S iff \( \forall \epsilon > 0 (S \cap (x_0 - \epsilon, x_0 + \epsilon) \setminus \{x_0\} \neq \emptyset) \)

X0 \( \in S \) is an isolated point of S iff \( \exists \epsilon > 0 (S \cap (x_0 - \epsilon, x_0 + \epsilon) \setminus \{x_0\} = \emptyset) \)

Example 5.1. If \( S = (0, 1] \), then \( S' = [0, 1] \) and S has no isolated points.

Example 5.2. If \( T = \{1/n : n \in \mathbb{Z} \setminus \{0\} \} \), then \( T' = \{0\} \) and all points of T are isolated points of T.

Theorem 5.6. \( x_0 \) is a limit point of S iff there is a sequence \( x_n \in S \setminus \{x_0\} \) such that \( x_n \to x_0 \).

Proof. \(( \Rightarrow ) \) For each \( n \in \mathbb{N} \) choose \( x_n \in S \cap (x_0 - 1/n, x_0 + 1/n) \setminus \{x_0\} \). Then \( |x_n - x_0| < 1/n \) for all \( n \in \mathbb{N} \), so \( x_n \to x_0 \).

\(( \Leftarrow ) \) Suppose \( x_n \) is a sequence from \( x_n \in S \setminus \{x_0\} \) converging to \( x_0 \). If \( \epsilon > 0 \), the definition of convergence for a sequence yields an \( N \in \mathbb{N} \) such that whenever \( n \geq N \), then \( x_n \in S \cap (x_0 - \epsilon, x_0 + \epsilon) \setminus \{x_0\} \). This shows \( S \cap (x_0 - \epsilon, x_0 + \epsilon) \setminus \{x_0\} \neq \emptyset \), and \( x_0 \) must be a limit point of S.

There is some common terminology making much of this easier to state. If \( x_0 \in \mathbb{R} \) and G is an open set containing \( x_0 \), then G is called a neighborhood of \( x_0 \). The observations given above can be restated in terms of neighborhoods.

Corollary 5.7. Let \( S \subseteq \mathbb{R} \).

(a) \( x_0 \) is a limit point of S iff every neighborhood of \( x_0 \) contains an infinite number of points from S.

(b) \( x_0 \in S \) is an isolated point of S iff there is a neighborhood of \( x_0 \) containing only a finite number of points from S.

Following is a generalization of Theorem 3.16.

Theorem 5.8 (Bolzano-Weierstrass Theorem). If \( S \subseteq \mathbb{R} \) is bounded and infinite, then \( S' \neq \emptyset \).

Proof. For the purposes of this proof, if \( I = [a, b] \) is a closed interval, let \( I^L = [a, (a + b)/2] \) be the closed left half of I and \( I^R = [(a + b)/2, b] \) be the closed right half of I.

Suppose S is a bounded and infinite set. The assumption that S is bounded implies the existence of an interval \( I_1 = [-B, B] \) containing S. Since S is infinite, at least one of the two sets \( I^L_1 \cap S \) or \( I^R_1 \cap S \) is infinite. Let \( I_2 \) be either \( I^L_1 \) or \( I^R_1 \) such that \( I_2 \cap S \) is infinite.

If \( I_n \) is such that \( I_n \cap S \) is infinite, let \( I_{n+1} \) be either \( I^L_n \) or \( I^R_n \), where \( I_{n+1} \cap S \) is infinite.

In this way, a nested sequence of intervals, \( I_n \) for \( n \in \mathbb{N} \), is defined such that \( I_n \cap S \) is infinite for all \( n \in \mathbb{N} \) and the length of \( I_n \) is \( B/2^n \to 0 \). According to the Nested Interval Theorem, there is an \( x_0 \in \mathbb{R} \) such that \( \bigcap_{n \in \mathbb{N}} I_n = \{x_0\} \).

To see \( x_0 \) is a limit point of S, for each \( n \), choose \( x_n \in S \cap I_n \setminus \{x_0\} \). This is possible because \( S \cap I_n \) is infinite. It follows that \( |x_n - x_0| < B/2^n \), so \( x_n \to x_0 \). Theorem 5.6 shows \( x_0 \in S' \).
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**Theorem 5.9.** A set \( S \subset \mathbb{R} \) is closed iff it contains all its limit points.

**Proof.** (\( \Rightarrow \)) Suppose \( S \) is closed and \( x_0 \) is a limit point of \( S \). If \( x_0 \notin S \), then \( S^c \) open implies the existence of \( \varepsilon > 0 \) such that \( (x_0 - \varepsilon, x_0 + \varepsilon) \cap S = \emptyset \). This contradicts the fact that \( x_0 \) is a limit point of \( S \). Therefore, \( x_0 \in S \), and \( S \) contains all its limit points.

(\( \Leftarrow \)) Since \( S \) contains all its limit points, if \( x_0 \notin S \), there must exist an \( \varepsilon > 0 \) such that \( (x_0 - \varepsilon, x_0 + \varepsilon) \cap S = \emptyset \). It follows from this that \( S^c \) is open. Therefore \( S \) is closed. \( \square \)

**Definition 5.10.** The closure of a set \( S \) is the set \( \overline{S} = S \cup S' \).

For the set \( S \) of Example 5.1, \( \overline{S} = [0, 1] \). In Example 5.2, \( \overline{T} = \{1/n : n \in \mathbb{Z} \setminus \{0\} \} \cup \{0\} \). According to Theorem 5.9, the closure of any set is a closed set. A useful way to think about this is that \( \overline{S} \) is the smallest closed set containing \( S \). This is made more precise in Exercise 5.2.

Following is a generalization of Theorem 3.20.

**Corollary 5.11.** If \( \{F_n : n \in \mathbb{N}\} \) is a nested collection of nonempty closed and bounded sets, then \( \bigcap_{n \in \mathbb{N}} F_n \neq \emptyset \).

**Proof.** Form a sequence \( x_n \) by choosing \( x_n \in F_n \) for each \( n \in \mathbb{N} \). Since the \( F_n \) are nested, \( \{x_n : n \in \mathbb{N}\} \subset F_1 \), and the boundedness of \( F_1 \) implies \( x_n \) is a bounded sequence. An application of Theorem 3.16 yields a subsequence \( y_n \) of \( x_n \) such that \( y_n \to y \). It suffices to prove \( y \in F_n \) for all \( n \in \mathbb{N} \).

To do this, fix \( n_0 \in \mathbb{N} \). Because \( y_n \) is a subsequence of \( x_n \) and \( x_{n_0} \in F_{n_0} \), it is easy to see \( y_n \in F_{n_0} \) for all \( n \geq n_0 \). Using the fact that \( y_n \to y \), we see \( y \in F_{n_0} \). Since \( F_{n_0} \) is closed, Theorem 5.9 shows \( y \in F_{n_0} \). \( \square \)

2. Relative Topologies and Connectedness

2.1. Relative Topologies. Another useful topological notion is that of a relative or subspace topology. In our case, this amounts to using the standard topology on \( \mathbb{R} \) to generate a topology on a subset of \( \mathbb{R} \). The definition is as follows.

**Definition 5.12.** Let \( X \subset \mathbb{R} \). The set \( S \subset X \) is relatively open in \( X \), if there is a set \( G \), open in \( \mathbb{R} \), such that \( S = G \cap X \). The set \( T \subset X \) is relatively closed in \( X \), if there is a set \( F \), closed in \( \mathbb{R} \), such that \( S = F \cap X \). (If there is no chance for confusion, the simpler terminology open in \( X \) and closed in \( X \) is sometimes used.)

It is left as exercises to show that if \( X \subset \mathbb{R} \) and \( \mathcal{T} \) consists of all relatively open subsets of \( X \), then \( (X, \mathcal{T}) \) is a topological space and \( T \) is relatively closed in \( X \), if \( X \setminus T \in \mathcal{T} \). (See Exercises 5.11 and 5.12.)

**Example 5.3.** Let \( X = [0, 1] \). The subsets \( [0, 1/2] = X \cap (-1, 1/2) \) and \( (1/4, 1) = X \cap (1/4, 2) \) are both relatively open in \( X \).

**Example 5.4.** If \( X = \mathbb{Q} \), then \( \{x \in \mathbb{Q} : -\sqrt{2} < x < \sqrt{2}\} = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q} = [-\sqrt{2}, \sqrt{2}] \cap \mathbb{Q} \) is clopen relative to \( \mathbb{Q} \).
2.2. Connected Sets. One place where the relative topologies are useful is in relation to the following definition.

**Definition 5.13.** A set \( S \subseteq \mathbb{R} \) is **disconnected** if there are two open intervals \( U \) and \( V \) such that \( U \cap V = \emptyset, U \cap S \neq \emptyset, V \cap S \neq \emptyset \) and \( S \subseteq U \cup V \). Otherwise, it is connected. The sets \( U \cap S \) and \( V \cap S \) are said to be a separation of \( S \).

In other words, \( S \) is disconnected if it can be written as the union of two disjoint and nonempty sets that are both relatively open in \( S \). Since both these sets are complements of each other relative to \( S \), they are both **clopen** in \( S \). This, in turn, implies \( S \) is disconnected if it has a proper relatively clopen subset.

**Example 5.5.** Let \( S = \{x\} \) be a set containing a single point. \( S \) is connected because there cannot exist nonempty disjoint open sets \( U \) and \( V \) such that \( S \cap U \neq \emptyset \) and \( S \cap V \neq \emptyset \). The same argument shows that \( \emptyset \) is connected.

**Example 5.6.** If \( S = [-1, 0) \cup (0, 1] \), then \( U = (-2, 0) \) and \( V = (0, 2) \) are open sets such that \( U \cap V = \emptyset, U \cap S \neq \emptyset, V \cap S \neq \emptyset \) and \( S \subseteq U \cup V \). This shows \( S \) is disconnected.

**Example 5.7.** The sets \( U = (-\infty, \sqrt{2}) \) and \( V = (\sqrt{2}, \infty) \) are open sets such that \( U \cap V = \emptyset, U \cap Q \neq \emptyset, V \cap Q \neq \emptyset \) and \( Q \subseteq U \cup V = \mathbb{R} \setminus \{\sqrt{2}\} \). This shows \( Q \) is disconnected. In fact, the only connected subsets of \( Q \) are single points. Sets with this property are often called **totally disconnected**.

The notion of connectedness is not really very interesting on \( \mathbb{R} \) because the connected sets are exactly what one would expect. It becomes more complicated in higher dimensional spaces.

**Theorem 5.14.** A nonempty set \( S \subseteq \mathbb{R} \) is connected iff it is either a single point or an interval.

**Proof.** (\( \Rightarrow \)) If \( S \) is not a single point or an interval, there must be numbers \( r < s < t \) such that \( r, t \in S \) and \( s \notin S \). In this case, the sets \( U = (-\infty, s) \) and \( V = (s, \infty) \) are a disconnection of \( S \).

(\( \Leftarrow \)) It was shown in Example 5.5 that a set containing a single point is connected. So, assume \( S \) is an interval.

Suppose \( S \) is not connected with \( U \) and \( V \) forming a disconnection of \( S \). Choose \( u \in U \cap S \) and \( v \in V \cap S \). There is no generality lost by assuming \( u < v \), so that \( [u, v] \subseteq S \).

Let \( A = \{x : [u, x) \subseteq U\} \).

We claim \( A \neq \emptyset \). To see this, use the fact that \( U \) is open to find \( \varepsilon \in (0, v - u) \) such that \( (u - \varepsilon, u + \varepsilon) \subseteq U \). Then \( u < u + \varepsilon/2 < v \), so \( u + \varepsilon/2 \in A \).

Define \( w = \text{lub} A \).

Since \( v \in V \) it is evident \( u < w \leq v \) and \( w \in S \).

If \( w \in U \), then \( u < w < v \) and there is \( \varepsilon \in (0, v - w) \) such that \( (w - \varepsilon, w + \varepsilon) \subseteq U \) and \( [u, w + \varepsilon] \subseteq S \) because \( w + \varepsilon < v \). This clearly contradicts the definition of \( w \), so \( w \notin U \).

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If \( w \in V \), then there is an \( \varepsilon > 0 \) such that \( (w - \varepsilon, w] \subset V \). In particular, this shows \( w = \text{lub} A \leq w - \varepsilon < w \). This contradiction forces the conclusion that \( w \notin V \).

Now, putting all this together, we see \( w \in S \subset U \cup V \) and \( w \notin U \cup V \). This is a clear contradiction, so we’re forced to conclude there is no separation of \( S \). \( \square \)

### 3. Covering Properties and Compactness on \( \mathbb{R} \)

#### 3.1. Open Covers.

**Definition 5.15.** Let \( S \subset \mathbb{R} \). A collection of open sets, \( \mathcal{O} = \{G_\lambda : \lambda \in \Lambda\} \), is an open cover of \( S \) if \( S \subset \bigcup_{G_\lambda \in \mathcal{O}} G \). If \( \mathcal{O}' \subset \mathcal{O} \) is also an open cover of \( S \), then \( \mathcal{O}' \) is an open subcover of \( S \) from \( \mathcal{O} \).

**Example 5.8.** Let \( S = (0, 1) \) and \( \mathcal{O} = \{(1/n, 1) : n \in \mathbb{N}\} \). We claim that \( \mathcal{O} \) is an open cover of \( S \). To prove this, let \( x \in (0, 1) \). Choose \( n_0 \in \mathbb{N} \) such that \( 1/n_0 < x \). Then

\[
x \in (1/n_0, 1) \subset \bigcup_{n \in \mathbb{N}} (1/n, 1) = \bigcup_{G_\lambda \in \mathcal{O}} G.
\]

Since \( x \) is an arbitrary element of \( (0, 1) \), it follows that \( (0, 1) = \bigcup_{G_\lambda \in \mathcal{O}} G \).

Suppose \( \mathcal{O}' \) is any infinite subset of \( \mathcal{O} \) and \( x \in (0, 1) \). Since \( \mathcal{O}' \) is infinite, there exists an \( n \in \mathbb{N} \) such that \( x \in (1/n, 1) \in \mathcal{O}' \). The rest of the proof proceeds as above.

On the other hand, if \( \mathcal{O}' \) is a finite subset of \( \mathcal{O} \), then let \( M = \max\{n : (1/n, 1) \in \mathcal{O}'\} \). If \( 0 < x < 1/M \), it is clear that \( x \notin \bigcup_{G_\lambda \in \mathcal{O}} G \), so \( \mathcal{O}' \) is not an open cover of \( (0, 1) \).

**Example 5.9.** Let \( T = [0, 1) \) and \( 0 < \varepsilon < 1 \). If

\[
\mathcal{O} = \{(1/n, 1) : n \in \mathbb{N}\} \cup \{(-\varepsilon, \varepsilon)\},
\]

then \( \mathcal{O} \) is an open cover of \( T \).

It is evident that any open subcover of \( T \) from \( \mathcal{O} \) must contain \( (-\varepsilon, \varepsilon) \), because that is the only element of \( \mathcal{O} \) which contains 0. Choose \( n \in \mathbb{N} \) such that \( 1/n < \varepsilon \). Then \( \mathcal{O}' = \{(-\varepsilon, \varepsilon), (1/n, 1)\} \) is an open subcover of \( T \) from \( \mathcal{O} \) which contains only two elements.

**Theorem 5.16 (Lindelöf Property).** If \( S \subset \mathbb{R} \) and \( \mathcal{O} \) is any open cover of \( S \), then \( \mathcal{O} \) contains a subcover with a countable number of elements.

**Proof.** Let \( \mathcal{O} = \{G_\lambda : \lambda \in \Lambda\} \) be an open cover of \( S \subset \mathbb{R} \). Since \( \mathcal{O} \) is an open cover of \( S \), for each \( x \in S \) there is a \( \lambda_x \in \Lambda \) and numbers \( p_x, q_x \in \mathbb{Q} \) satisfying \( x \in (p_x, q_x) \subset G_{\lambda_x} \in \mathcal{O} \). The collection \( \mathcal{T} = \{(p_x, q_x) : x \in S\} \) is an open cover of \( S \).

Thinking of the collection \( \mathcal{T} = \{(p_x, q_x) : x \in S\} \) as a set of ordered pairs of rational numbers, it is seen that \( \text{card} (\mathcal{T}) \leq \text{card} (\mathbb{Q} \times \mathbb{Q}) = \aleph_0 \), so \( \mathcal{T} \) is countable.

For each interval \( I \in \mathcal{T} \), choose a \( \lambda_I \in \Lambda \) such that \( I \subset G_{\lambda_I} \). Then

\[
S \subset \bigcup_{I \in \mathcal{T}} I \subset \bigcup_{I \in \mathcal{T}} G_{\lambda_I},
\]

shows \( \mathcal{O}' = \{G_{\lambda_I} : I \in \mathcal{T}\} \subset \mathcal{O} \) is an open subcover of \( S \) from \( \mathcal{O} \). Also, \( \text{card} (\mathcal{O}') \leq \text{card} (\mathcal{T}) \leq \aleph_0 \), so \( \mathcal{O}' \) is a countable open subcover of \( S \) from \( \mathcal{O} \). \( \square \)
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**Corollary 5.17.** Any open subset of \( \mathbb{R} \) can be written as a countable union of pairwise disjoint open intervals.

**Proof.** Let \( G \) be open in \( \mathbb{R} \). For \( x \in G \) let \( \alpha_x = \text{glb} \{y : (y, x] \subset G\} \) and \( \beta_x = \text{lub} \{y : [x, y) \subset G\} \). The fact that \( G \) is open implies \( \alpha_x < x < \beta_x \). Define \( I_x = (\alpha_x, \beta_x) \).

Then \( I_x \subset G \). To see this, suppose \( x < w < \beta_x \). Choose \( y \in (w, \beta_x) \). The definition of \( \beta_x \) guarantees \( w \in (x, y) \subset G \). Similarly, if \( \alpha_x < w < x \), it follows that \( w \in G \).

This shows \( \mathcal{O} = \{I_x : x \in G\} \) has the property that \( G = \bigcup_{x \in G} I_x \).

Suppose \( x, y \in G \) and \( I_x \cap I_y \neq \emptyset \). There is no generality lost in assuming \( x < y \). In this case, there must be a \( w \in (x, y) \) such that \( w \in I_x \cap I_y \). We know from above that both \( [x, w] \subset G \) and \( [w, y] \subset G \), so \( [x, y] \subset G \). It follows that \( \alpha_x < x < y < \beta_x = \beta_y \) and \( I_x = I_y \).

From this we conclude \( \mathcal{O} \) consists of pairwise disjoint open intervals.

To finish, apply Theorem 5.16 to extract a countable subcover from \( \mathcal{O} \). \( \square \)

Corollary 5.17 can also be proved by a different strategy. Instead of using Theorem 5.16 to extract a countable subcover, we could just choose one rational number from each interval in the cover. The pairwise disjointness of the intervals in the cover guarantee this will give a bijection between \( \mathcal{O} \) and a subset of \( \mathbb{Q} \). This method has the advantage of showing \( \mathcal{O} \) itself is countable from the start.

### 3.2. Compact Sets

There is a class of sets for which the conclusion of Linderöf’s theorem can be strengthened.

**Definition 5.18.** An open cover \( \mathcal{O} \) of a set \( S \) is a **finite cover**, if \( \mathcal{O} \) has only a finite number of elements. The definition of a **finite subcover** is analogous.

**Definition 5.19.** A set \( K \subset \mathbb{R} \) is **compact**, if every open cover of \( K \) contains a finite subcover.

**Theorem 5.20** (Heine-Borel). A set \( K \subset \mathbb{R} \) is compact iff it is closed and bounded.

**Proof.** \((\Rightarrow)\) Suppose \( K \) is unbounded. The collection \( \mathcal{O} = \{(-n, n) : n \in \mathbb{N}\} \) is an open cover of \( K \). If \( \mathcal{O}' \) is any finite subcover of \( \mathcal{O} \), then \( \bigcup_{G \in \mathcal{O}'} G \) is a bounded set and cannot cover the unbounded set \( K \). This shows \( K \) cannot be compact, and every compact set must be bounded.

Suppose \( K \) is not closed. According to Theorem 5.9, there is a limit point \( x \) of \( K \) such that \( x \notin K \). Define \( \mathcal{O} = \{[x - 1/n, x + 1/n] : n \in \mathbb{N}\} \). Then \( \mathcal{O} \) is a collection of open sets and \( K \subset \bigcup_{G \in \mathcal{O}} G = \mathbb{R} \setminus \{x\} \). Let \( \mathcal{O}' = \{[x - 1/n_i, x + 1/n_i] : 1 \leq i \leq N\} \) be a finite subcover of \( \mathcal{O} \) and \( M = \max(n_i : 1 \leq i \leq N) \). Since \( x \) is a limit point of \( K \), there is a \( y \in K \cap (x - 1/M, x + 1/M) \). Clearly, \( y \notin \bigcup_{G \in \mathcal{O}'} G = [x - 1/M, x + 1/M] \), so \( \mathcal{O}' \) cannot cover \( K \). This shows every compact set must be closed.

\((\Leftarrow)\) Let \( K \) be closed and bounded and let \( \mathcal{O} \) be an open cover of \( K \). Applying Theorem 5.16, if necessary, we can assume \( \mathcal{O} \) is countable. Thus, \( \mathcal{O} = \{G_n : n \in \mathbb{N}\} \).
For each $n \in \mathbb{N}$, define $$F_n = K \setminus \bigcup_{i=1}^{n} G_i = K \cap \bigcap_{i=1}^{n} G_i^c.$$ Then $F_n$ is a sequence of nested, bounded and closed subsets of $K$. Since $\emptyset$ covers $K$, it follows that $$\bigcap_{n \in \mathbb{N}} F_n \subset K \setminus \bigcup_{n \in \mathbb{N}} G_n = \emptyset.$$

According to the Corollary 5.11, the only way this can happen is if $F_n = \emptyset$ for some $n \in \mathbb{N}$. Then $K \setminus \bigcup_{i=1}^{n} G_i = \emptyset$. According to the Corollary 5.11, the only way this can happen is if $F_n = \emptyset$ for some $n \in \mathbb{N}$. Then $K \setminus \bigcup_{i=1}^{n} G_i = \emptyset$, and $\emptyset' = \{G_i : 1 \leq i \leq n\}$ is a finite subcover of $K$ from $\emptyset$.

Compactness shows up in several different, but equivalent ways on $\mathbb{R}$. We’ve already seen several of them, but their equivalence is not obvious. The following theorem shows a few of the most common manifestations of compactness.

**Theorem 5.21.** Let $K \subset \mathbb{R}$. The following statements are equivalent to each other.

(a) $K$ is compact.
(b) $K$ is closed and bounded.
(c) Every infinite subset of $K$ has a limit point in $K$.
(d) Every sequence $\{a_n : n \in \mathbb{N}\} \subset K$ has a subsequence converging to an element of $K$.
(e) If $F_n$ is a nested sequence of nonempty relatively closed subsets of $K$, then $\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset$.

**Proof.** (a) $\iff$ (b) is the Heine-Borel Theorem, Theorem 5.20. That (b)$\implies$(c) is the Bolzano-Weierstrass Theorem, Theorem 5.8. (c)$\implies$(d) is contained in the sequence version of the Bolzano-Weierstrass theorem, Theorem 3.16.

(d)$\implies$(e) Let $F_n$ be as in (e). For each $n \in \mathbb{N}$, choose $a_n \in F_n$. By assumption, $a_n$ has a convergent subsequence $b_n \rightarrow b$. Each $F_n$ contains a tail of the sequence $b_n$, so $b \in F_n' \subset F_n$ for each $n$. Therefore, $b \in \bigcap_{n \in \mathbb{N}} F_n$, and (e) follows.

(e)$\implies$(b). Suppose $K$ is such that (e) is true. Let $F_n = K \cap ((-\infty, -n] \cup [n, \infty))$. Then $F_n$ is a sequence of sets which are relatively closed in $K$ such that $\bigcap_{n \in \mathbb{N}} F_n = \emptyset$. If $K$ is unbounded, then $F_n \neq \emptyset$, $\forall n \in \mathbb{N}$, and a contradiction of (e) is evident. Therefore, $K$ must be bounded. If $K$ is not closed, then there must be a limit point $x$ of $K$ such that $x \notin K$. Define a sequence of relatively closed and nested subsets of $K$ by $F_n = [x-1/n, x+1/n] \cap K$ for $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} F_n = \emptyset$, because $x \notin K$. This contradiction of (e) shows that $K$ must be closed.

These various ways of looking at compactness have been given different names by topologists. Property (c) is called limit point compactness and (d) is called sequential compactness. There are topological spaces in which various of the equivalences do not hold.
4. More Small Sets

This is an advanced section that can be omitted.

We’ve already seen one way in which a subset of \( \mathbb{R} \) can be considered small—if its cardinality is at most \( \aleph_0 \). Such sets are small in the set-theoretic sense. This section shows how sets can be considered small in the metric and topological senses.

4.1. Sets of Measure Zero. An interval is the only subset of \( \mathbb{R} \) for which most people could immediately come up with some sort of measure — its length. This idea of measuring a set by length can be generalized. For example, we know every open set can be written as a countable union of open intervals, so it is natural to assign the sum of the lengths of its component intervals as the measure of the set. Discounting some technical difficulties, such as components with infinite length, this is how the Lebesgue measure of an open set is defined. It is possible to assign a measure to more complicated sets, but we’ll only address the special case of sets with measure zero, sometimes called Lebesgue null sets.

**Definition 5.22.** A set \( S \subseteq \mathbb{R} \) has measure zero if given any \( \varepsilon > 0 \) there is a sequence \((a_n, b_n)\) of open intervals such that

\[
S \subseteq \bigcup_{n \in \mathbb{N}} (a_n, b_n) \quad \text{and} \quad \sum_{n=1}^{\infty} (b_n - a_n) < \varepsilon.
\]

Such sets are small in the metric sense.

**Example 5.10.** If \( S = \{a\} \) contains only one point, then \( S \) has measure zero. To see this, let \( \varepsilon > 0 \). Note that \( S \subset (a - \varepsilon/4, a + \varepsilon/4) \) and this single interval has length \( \varepsilon/2 < \varepsilon \).

There are complicated sets of measure zero, as we’ll see later. For now, we’ll start with a simple theorem.

**Theorem 5.23.** If \( \{S_n : n \in \mathbb{N}\} \) is a countable collection of sets of measure zero, then \( \bigcup_{n \in \mathbb{N}} S_n \) has measure zero.

**Proof.** Let \( \varepsilon > 0 \). For each \( n \), let \( \{(a_{n,k}, b_{n,k}) : k \in \mathbb{N}\} \) be a collection of intervals such that

\[
S_n \subset \bigcup_{k \in \mathbb{N}} (a_{n,k}, b_{n,k}) \quad \text{and} \quad \sum_{k=1}^{\infty} (b_{n,k} - a_{n,k}) < \frac{\varepsilon}{2^n}.
\]

Then

\[
\bigcup_{n \in \mathbb{N}} S_n \subset \bigcup_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} (a_{n,k}, b_{n,k}) \quad \text{and} \quad \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (b_{n,k} - a_{n,k}) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.
\]

Combining this with Example 5.10 gives the following corollary.

**Corollary 5.24.** Every countable set has measure zero.
The rational numbers is a large set in the sense that every interval contains a rational number. But we now see it is small in both the set theoretic and metric senses because it is countable and of measure zero.

Uncountable sets of measure zero are constructed in Section 4.3.

There is some standard terminology associated with sets of measure zero. If a property $P$ is true, except on a set of measure zero, then it is often said "$P$ is true almost everywhere" or "almost every point satisfies $P."" It is also said "$P$ is true on a set of full measure." For example, "Almost every real number is irrational," or "The irrational numbers are a set of full measure."

### 4.2. Dense and Nowhere Dense Sets

We begin by considering a way that a set can be considered topologically large in an interval. If $I$ is any interval, recall from Corollary 2.24 that $I \cap \mathbb{Q} \neq \emptyset$ and $I \cap \mathbb{Q}^c \neq \emptyset$. An immediate consequence of this is that every real number is a limit point of both $\mathbb{Q}$ and $\mathbb{Q}^c$. In this sense, the rational and irrational numbers are both uniformly distributed across the number line. This idea is generalized in the following definition.

**Definition 5.25.** Let $A \subset B \subset \mathbb{R}$. A is said to be dense in $B$, if $B \subset \overline{A}$.

Both the rational and irrational numbers are dense in every interval. Corollary 5.17 shows the rational and irrational numbers are dense in every open set. It’s not hard to construct other sets dense in every interval. For example, the set of dyadic numbers, $\mathcal{D} = \{p/2^q : p, q \in \mathbb{Z}\}$, is dense in every interval — and dense in the rational numbers.

On the other hand, $\mathbb{Z}$ is not dense in any interval because it’s closed and contains no interval. If $A \subset B$, where $B$ is an open set, then $A$ is not dense in $B$, if $A$ contains any interval-sized gaps.

**Theorem 5.26.** Let $A \subset B \subset \mathbb{R}$. $A$ is dense in $B$ iff whenever $I$ is an open interval such that $I \cap B \neq \emptyset$, then $I \cap A \neq \emptyset$.

**Proof.** ($\Rightarrow$) Assume there is an open interval $I$ such that $I \cap B \neq \emptyset$ and $I \cap A = \emptyset$. If $x \in I \cap B$, then $I$ is a neighborhood of $x$ that does not intersect $A$. Definition 5.5 shows $x \notin A^c \subset \overline{A}$, a contradiction of the assumption that $B \subset \overline{A}$. This contradiction implies that whenever $I \cap B \neq \emptyset$, then $I \cap A \neq \emptyset$.

($\Leftarrow$) If $x \in B \cap A = A$, then $x \in \overline{A}$. Assume $x \in B \setminus A$. By assumption, for each $n \in \mathbb{N}$, there is an $x_n \in (x - 1/n, x + 1/n) \cap A$. Since $x_n \to x$, this shows $x \in A^c \subset \overline{A}$. It now follows that $B \subset \overline{A}$. \qed

If $B \subset \mathbb{R}$ and $I$ is an open interval with $I \cap B \neq \emptyset$, then $I \cap B$ is often called a portion of $B$. The previous theorem says that $A$ is dense in $B$ iff every portion of $B$ intersects $A$.

If $A$ being dense in $B$ is thought of as $A$ being a large subset of $B$, then perhaps when $A$ is not dense in $B$, it can be thought of as a small subset. But, thinking of $A$ as being small when it is not dense isn’t quite so clear when it is noticed that $A$ could still be dense in some portion of $B$, even if it isn’t dense in $B$. To make $A$ be a truly small subset of $B$ in the topological sense, it should not be dense in any portion of $B$. The following definition gives a way to assure this is true.
DEFINITION 5.27. Let \( \Omega \subset B \subset \mathbb{R} \). \( \Omega \) is said to be **nowhere dense** in \( B \) if \( B \setminus \overline{\Omega} \) is dense in \( B \).

The following theorem shows that a nowhere dense set is small in the sense mentioned above because it fails to be dense in any part of \( B \).

**Theorem 5.28.** Let \( \Omega \subset B \subset \mathbb{R} \). \( \Omega \) is nowhere dense in \( B \) iff for every open interval \( I \) such that \( I \cap B \neq \emptyset \), there is an open interval \( J \subset I \) such that \( J \cap B \neq \emptyset \) and \( J \cap \Omega = \emptyset \).

**Proof.** (\( \Rightarrow \)) Let \( I \) be an open interval such that \( I \cap B \neq \emptyset \). By assumption, \( B \setminus \overline{\Omega} \) is dense in \( B \), so Theorem 5.26 implies \( I \cap (B \setminus \overline{\Omega}) \neq \emptyset \). If \( x \in I \cap (B \setminus \overline{\Omega}) \), then there is an open interval \( J \) such that \( x \in J \subset I \) and \( J \cap \overline{\Omega} = \emptyset \). Since \( \Omega \subset \overline{\Omega} \), this \( J \) satisfies the theorem.

(\( \Leftarrow \)) Let \( I \) be an open interval with \( I \cap B \neq \emptyset \). By assumption, there is an open interval \( J \subset I \) such that \( J \cap \Omega = \emptyset \). It follows that \( J \cap \overline{\Omega} = \emptyset \). Theorem 5.26 implies \( B \setminus \overline{\Omega} \) is dense in \( B \).

**Example 5.11.** Let \( G \) be an open set that is dense in \( \mathbb{R} \). If \( I \) is any open interval, then Theorem 5.26 implies \( I \setminus G \neq \emptyset \). Because \( G \) is open, if \( x \in I \setminus G \), then there is an open interval \( J \) such that \( x \in J \subset G \). Now, Theorem 5.28 shows \( G^c \) is nowhere dense.

The nowhere dense sets are topologically small in the following sense.

**Theorem 5.29 (Baire).** If \( I \) is an open interval, then \( I \) cannot be written as a countable union of nowhere dense sets.

**Proof.** Let \( A_n \) be a sequence of nowhere dense subsets of \( I \). According to Theorem 5.28, there is a bounded open interval \( J_1 \subset I \) such that \( J_1 \cap A_1 = \emptyset \). By shortening \( J_1 \) a bit at each end, if necessary, it may be assumed that \( J_1 \cap A_1 = \emptyset \). Assume \( J_n \) has been chosen for some \( n \in \mathbb{N} \). Applying Theorem 5.28 again, choose an open interval \( J_{n+1} \) as above so \( J_{n+1} \subset J_n \) and \( J_{n+1} \cap A_{n+1} = \emptyset \). Corollary 5.11 shows

\[
I \setminus \bigcup_{n \in \mathbb{N}} A_n \supseteq \bigcap_{n \in \mathbb{N}} J_n \neq \emptyset
\]

and the theorem follows.

Theorem 5.29 is called the **Baire category theorem** because of the terminology introduced by René-Louis Baire in 1899.\(^3\) He said a set was of the first category, if it could be written as a countable union of nowhere dense sets. An easy example of such a set is any countable set, which is a countable union of singletons. All other sets are of the second category.\(^4\) Theorem 5.29 can be stated as “Any open interval is of the second category.” Or, more generally, as “Any nonempty open set is of the second category.”

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\(^3\)René-Louis Baire (1874-1932) was a French mathematician. He proved the Baire category theorem in his 1899 doctoral dissertation.

\(^4\)Baire did not define any categories other than these two. Some authors call first category sets meager sets, so as not to make readers fruitlessly wait for definitions of third, fourth and fifth category sets.
A set is called a $G_δ$ set, if it is the countable intersection of open sets. It is called an $F_σ$ set, if it is the countable union of closed sets. DeMorgan’s laws show that the complement of an $F_σ$ set is a $G_δ$ set and vice versa. It’s evident that any countable subset of $\mathbb{R}$ is an $F_σ$ set, so $\mathbb{Q}$ is an $F_σ$ set.

On the other hand, suppose $\mathbb{Q}$ is a $G_δ$ set. Then there is a sequence of open sets $G_n$ such that $\mathbb{Q} = \bigcap_{n \in \mathbb{N}} G_n$. Since $\mathbb{Q}$ is dense, each $G_n$ must be dense and Example 5.11 shows $G_n^c$ is nowhere dense. From DeMorgan’s law, $\mathbb{R} = \mathbb{Q} \cup \bigcup_{n \in \mathbb{N}} G_n^c$, showing $\mathbb{R}$ is a first category set and violating the Baire category theorem. Therefore, $\mathbb{Q}$ is not a $G_δ$ set.

Essentially the same argument shows any countable subset of $\mathbb{R}$ is a first category set. The following protracted example shows there are uncountable sets of the first category.

### 4.3. The Cantor Middle-Thirds Set.

One particularly interesting example of a nowhere dense set is the Cantor Middle-Thirds set, introduced by the German mathematician Georg Cantor in 1884.\textsuperscript{5} It has many strange properties, only a few of which will be explored here.

To start the construction of the Cantor Middle-Thirds set, let $C_0 = [0, 1]$ and $C_1 = I_1 \setminus (1/3, 2/3) = [0, 1/3] \cup [2/3, 1]$. Remove the open middle thirds of the intervals comprising $C_1$, to get

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

Continuing in this way, if $C_n$ consists of $2^n$ pairwise disjoint closed intervals each of length $3^{-n}$, construct $C_{n+1}$ by removing the open middle third from each of those closed intervals, leaving $2^{n+1}$ closed intervals each of length $3^{-(n+1)}$. This gives a nested sequence of closed sets $C_n$, each consisting of $2^n$ closed intervals of length $3^{-n}$. (See Figure 5.1.) The Cantor Middle-Thirds set is

$$C = \bigcap_{n \in \mathbb{N}} C_n.$$

\textbf{Figure 5.1.} Shown here are the first few steps in the construction of the Cantor Middle-Thirds set.

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\textsuperscript{5}Cantor’s original work [6] is reprinted with an English translation in Edgar’s *Classics on Fractals* [11]. Cantor only mentions his eponymous set in passing and it had actually been presented earlier by others.
Corollaries 5.3 and 5.11 show $C$ is closed and nonempty. In fact, the latter is apparent because $\{0, 1/3, 2/3, 1\} \subset C_n$ for every $n$. At each step in the construction, $2^n$ open middle thirds, each of length $3^{-(n+1)}$ were removed from the intervals comprising $C_n$. The total length of the open intervals removed was

$$\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1.$$ 

Because of this, Example 5.11 implies $C$ is nowhere dense in $[0, 1]$.

$C$ is an example of a perfect set; i.e., a closed set all of whose points are limit points of itself. (See Exercise 5.24.) Any closed set without isolated points is perfect. The Cantor Middle-Thirds set is interesting because it is an example of a perfect set without any interior points. Many people call any bounded perfect set without interior points a Cantor set. Most of the time, when someone refers to the Cantor set, they mean $C$.

There is another way to view the Cantor set. Notice that at the $n$th stage of the construction, removing the middle thirds of the intervals comprising $C_n$ removes those points whose base 3 representation contains the digit 1 in position $n + 1$. Then,

$$C = \left\{ c = \sum_{n=1}^{\infty} \frac{c_n}{3^n} : c_n \in \{0, 2\} \right\}. \tag{5.1}$$

So, $C$ consists of all numbers $c \in [0, 1]$ that can be written in base 3 without using the digit 1.\footnote{Notice that $1 = \sum_{n=1}^{\infty} \frac{2}{3^n}$, $1/3 = \sum_{n=2}^{\infty} \frac{2}{3^n}$, etc.}

If $c \in C$, then (5.1) shows $c = \sum_{n=1}^{\infty} \frac{c_n}{3^n}$ for some sequence $c_n$ with range in $\{0, 2\}$. Moreover, every such sequence corresponds to a unique element of $C$. Define $\phi : C \to [0, 1]$ by

$$\phi(c) = \phi \left( \sum_{n=1}^{\infty} \frac{c_n}{3^n} \right) = \sum_{n=1}^{\infty} \frac{c_n/2}{2^n}. \tag{5.2}$$

Since $c_n$ is a sequence from $\{0, 2\}$, then $c_n/2$ is a sequence from $\{0, 1\}$ and $\phi(c)$ can be considered the binary representation of a number in $[0, 1]$. According to (5.1), it follows that $\phi$ is a surjection and

$$\phi(C) = \left\{ \sum_{n=1}^{\infty} \frac{c_n/2}{2^n} : c_n \in \{0, 2\} \right\} = \left\{ \sum_{n=1}^{\infty} \frac{b_n}{2^n} : b_n \in \{0, 1\} \right\} = [0, 1].$$

Therefore, $\text{card}(C) = \text{card}([0, 1]) > \aleph_0$.

The Cantor set is topologically small because it is nowhere dense and large from the set-theoretic viewpoint because it is uncountable.

The Cantor set is also a set of measure zero. To see this, let $C_n$ be as in the construction of the Cantor set given above. Then $C \subset C_n$ and $C_n$ consists of $2^n$ pairwise disjoint closed intervals each of length $3^{-n}$. Their total length is $(2/3)^n$. Given $\varepsilon > 0$, choose $n \in \mathbb{N}$ so $(2/3)^n < \varepsilon/2$. Each of the closed intervals comprising $C_n$ can be placed inside a slightly longer open interval so the sums of the lengths of the $2^n$ open intervals is less than $\varepsilon$.\footnote{Notice that $1 = \sum_{n=1}^{\infty} 2/3^n$, $1/3 = \sum_{n=2}^{\infty} 2/3^n$, etc.}
5. Exercises

5.1. If \( G \) is an open set and \( F \) is a closed set, then \( G \setminus F \) is open and \( F \setminus G \) is closed.

5.2. Let \( S \subseteq \mathbb{R} \) and \( \mathcal{F} = \{ F : F \) is closed and \( S \subseteq F \} \). Prove \( \overline{S} = \bigcap_{F \in \mathcal{F}} F \). This proves that \( \overline{S} \) is the smallest closed set containing \( S \).

5.3. Let \( S \) and \( T \) be subsets of \( \mathbb{R} \). Prove or give a counterexample:
   (a) \( \overline{A \cup B} = \overline{A} \cup \overline{B} \), and
   (b) \( \overline{A \cap B} = \overline{A} \cap \overline{B} \)

5.4. If \( S \) is a finite subset of \( \mathbb{R} \), then \( S \) is closed.

5.5. For any sets \( A, B \subseteq \mathbb{R} \), define
   \[ A + B = \{ a + b : a \in A \text{ and } b \in B \} \]
   (a) If \( X, Y \subseteq \mathbb{R} \), then \( \overline{X + Y} \subseteq \overline{X} + \overline{Y} \).
   (b) Find an example to show equality may not hold in the preceding statement.

5.6. \( \emptyset \) is neither open nor closed.

5.7. A set \( S \subseteq \mathbb{R} \) is open iff \( \partial S \cap S = \emptyset \). (\( \partial S \) is the set of boundary points of \( S \).)

5.8. Find a sequence of open sets \( G_n \) such that \( \bigcap_{n \in \mathbb{N}} G_n \) is neither open nor closed.

5.9. An open set \( G \) is called regular if \( G = (\overline{G})^c \). Find an open set that is not regular.

5.10. Let \( \mathcal{R} = \{(x, \infty) : x \in \mathbb{R} \} \) and \( \mathcal{T} = \mathcal{R} \cup \{ \mathbb{R}, \emptyset \} \). Prove that \( (\mathbb{R}, \mathcal{T}) \) is a topological space. This is called the right ray topology on \( \mathbb{R} \).

5.11. If \( X \subseteq \mathbb{R} \) and \( \mathcal{S} \) is the collection of all sets relatively open in \( X \), then \( (X, \mathcal{S}) \) is a topological space.

5.12. If \( X \subseteq \mathbb{R} \) and \( G \) is an open set, then \( X \setminus G \) is relatively closed in \( X \).

5.13. For any set \( S \), let \( \mathcal{F} = \{ T \subseteq S : \text{card}(S \setminus T) \leq \aleph_0 \} \cup \{ \emptyset \} \). Then \( (S, \mathcal{F}) \) is a topological space. This is called the finite complement topology.

5.14. An uncountable subset of \( \mathbb{R} \) must have a limit point.

5.15. If \( S \subseteq \mathbb{R} \), then \( S' \) is closed.

5.16. Prove that the set of accumulation points of any sequence is closed.

5.17. Prove any closed set is the set of accumulation points for some sequence.
5.18. If \( a_n \) is a sequence such that \( a_n \to L \), then \{ \( a_n : n \in \mathbb{N} \) \} \cup \{L\} is compact.

5.19. If \( F \) is closed and \( K \) is compact, then \( F \cap K \) is compact.

5.20. If \( \{K_\alpha : \alpha \in A\} \) is a collection of compact sets, then \( \bigcap_{\alpha \in A} K_\alpha \) is compact.

5.21. Prove the union of a finite number of compact sets is compact. Give an example to show this need not be true for the union of an infinite number of compact sets.

5.22. (a) Give an example of a set \( S \) such that \( S \) is disconnected, but \( S \cup \{1\} \) is connected. (b) Prove that 1 must be a limit point of \( S \).

5.23. If \( K \) is compact and \( V \) is open with \( K \subset V \), then there is an open set \( U \) such that \( K \subset U \subset \overline{U} \subset V \).

5.24. If \( C \) is the Cantor middle-thirds set, then \( C = C' \).

5.25. If \( x \in \mathbb{R} \) and \( K \) is compact, then there is a \( z \in K \) such that \( |x - z| = \text{glb}\{|x - y| : y \in K\} \). Is \( z \) unique?

5.26. If \( K \) is compact and \( \mathcal{O} \) is an open cover of \( K \), then there is an \( \epsilon > 0 \) such that for all \( x \in K \) there is a \( G \in \mathcal{O} \) with \( (x - \epsilon, x + \epsilon) \subset G \).

5.27. Let \( f : [a, b] \to \mathbb{R} \) be a function such that for every \( x \in [a, b] \) there is a \( \delta_x > 0 \) such that \( f \) is bounded on \( (x - \delta_x, x + \delta_x) \). Prove \( f \) is bounded.

5.28. Is the function defined by (5.2) a bijection?

5.29. If \( A \) is nowhere dense in an interval \( I \), then \( \overline{A} \) contains no interval.

5.30. Use the Baire category theorem to show \( \mathbb{R} \) is uncountable.