Chapter 7

Differentiation

7.1 The Derivative at a Point

Definition 7.1.1. Let f be a function on a neighborhood of x_0. f is *differentiable* at x_0, if the following limit exists:

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

Define $D(f) = \{x : f'(x) \text{ exists}\}$.

The standard notations for the derivative will be used; e. g., $f'(x)$, $\frac{df(x)}{dx}$, $Df(x)$, etc.

An equivalent way of stating this definition is to note that if $x_0 \in D(f)$, then

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

This can be interpreted in the standard way as the limiting slope of the secant line as the points of intersection approach each other.

Example 7.1.1. If $f(x) = c$ for all x and some $c \in \mathbb{R}$, then

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{c - c}{h} = 0.$$

So, $f'(x) = 0$ everywhere.

Example 7.1.2. If $f(x) = x$, then

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{x_0 + h - x_0}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

So, $f'(x) = 1$ everywhere.

[1]May 1, 2009

Theorem 7.1.1. *For any function f, $D(f) \subset C(f)$.*

Proof. Suppose $x_0 \in D(f)$. Then

$$\lim_{x \to x_0} |f(x) - f(x_0)| = \lim_{x \to x_0} \left| \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \right| = f'(x_0)\ 0 = 0.$$

This shows $\lim_{x \to x_0} f(x) = f(x_0)$, and $x_0 \in C(f)$. □

Of course, the converse of Theorem 7.1.1 is not true.

Example 7.1.3. The function $f(x) = |x|$ is continuous on $\mathbb{R}$, but

$$\lim_{h \downarrow 0} \frac{f(0+h) - f(0)}{h} = 1 = -\lim_{h \uparrow 0} \frac{f(0+h) - f(0)}{h},$$

so $f'(0)$ fails to exist.

Theorem 7.1.1 and Example 7.1.3 show that differentiability is a strictly stronger condition than continuity. For a long time most mathematicians thought that every continuous function must certainly be differentiable at some point. In 1872, Karl Weierstrass presented a function continuous on $\mathbb{R}$ which is differentiable nowhere.[2] It has since been proved that the "typical" continuous function is nowhere differentiable. So, contrary to the impression left by many beginning calculus classes, differentiability is the exception rather than the rule.

The next few theorems contain the standard differentiation rules.

Theorem 7.1.2. *Suppose f and g are functions such that $x_0 \in D(f) \cap D(g)$.*

(a) $x_0 \in D(f+g)$ and $(f+g)'(x_0) = f'(x_0) + g'(x_0)$.

(b) If $a \in \mathbb{R}$, then $x_0 \in D(af)$ and $(af)'(x_0) = af'(x_0)$.

(c) $x_0 \in D(fg)$ and $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

(d) If $g(x_0) \neq 0$, then $x_0 \in D(f/g)$ and

$$\left(\frac{f}{g}\right)(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}.$$

Proof. (a)

$$\begin{aligned}\lim_{h \to 0} & \frac{(f+g)(x_0+h) - (f+g)(x_0)}{h} \\ &= \lim_{h \to 0} \frac{f(x_0+h) + g(x_0+h) - f(x_0) - g(x_0)}{h} \\ &= \lim_{h \to 0} \left(\frac{f(x_0+h) - f(x_0)}{h} + \frac{g(x_0+h) - g(x_0)}{h} \right) = f'(x_0) + g'(x_0)\end{aligned}$$

[2] A translation of Weierstrass' original paper [6] is presented by Edgar [2]. Weierstrass' example is not very transparent because it depends on trigonometric series. Many more elementary constructions have since been made. One such will be presented in Example 9.5.1.

(b)

$$\lim_{h\to 0}\frac{(af)(x_0+h)-(af)(x_0)}{h}=a\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=af'(x_0)$$

(c)

$$\lim_{h\to 0}\frac{(fg)(x_0+h)-(fg)(x_0)}{h}=\lim_{h\to 0}\frac{f(x_0+h)g(x_0+h)-f(x_0)g(x_0)}{h}$$

Now, "slip a 0" into the numerator and factor the fraction.

$$\begin{aligned}&=\lim_{h\to 0}\frac{f(x_0+h)g(x_0+h)-f(x_0)g(x_0+h)+f(x_0)g(x_0+h)-f(x_0)g(x_0)}{h}\\ &\qquad=\lim_{h\to 0}\left(\frac{f(x_0+h)-f(x_0)}{h}g(x_0+h)+f(x_0)\frac{g(x_0+h)-g(x_0)}{h}\right)\end{aligned}$$

Finally, use the definition of the derivative and the continuity of f and g at x_0.

$$=f'(x_0)g(x_0)+f(x_0)g'(x_0)$$

(d) It will be proved that if $g(x_0)\neq 0$, then $(1/g)'(x_0)=-g'(x_0)/(g(x_0))^2$. This statement, combined with (c), yields (d).

$$\begin{aligned}\lim_{h\to 0}\frac{(1/g)(x_0+h)-(1/g)(x_0)}{h}&=\lim_{h\to 0}\frac{\dfrac{1}{g(x_0+h)}-\dfrac{1}{g(x_0)}}{h}\\ &=\lim_{h\to 0}\frac{g(x_0)-g(x_0+h)}{h}\frac{1}{g(x_0+h)g(x_0)}\\ &=-\frac{g'(x_0)}{(g(x_0)^2}\end{aligned}$$

Plug this into (c) to see

$$\begin{aligned}\left(\frac{f}{g}\right)'(x_0)&=\left(f\frac{1}{g}\right)'(x_0)\\ &=f'(x_0)\frac{1}{g(x_0)}+f(x_0)\frac{-g'(x_0)}{(g(x_0))^2}\\ &=\frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{(g(x_0))^2}.\end{aligned}$$

□

Combining Examples 7.1.1 and 7.1.2 with Theorem 7.1.2, the following theorem is easy to prove.

Theorem 7.1.3. *A rational function is differentiable at every point of its domain.*

Theorem 7.1.4 (Chain Rule)**.** *If f and g are functions such that $x_0 \in D(f)$ and $f(x_0) \in D(g)$, then $x_0 \in D(g \circ f)$ and $(g \circ f)'(x_0) = g' \circ f(x_0)f'(x_0)$.*

Proof. Let $y_0 = f(x_0)$. By assumption, there is an open interval J containing $f(x_0)$ such that g is defined on J. Since J is open and $x_0 \in C(f)$, there is an open interval I containing x_0 such that $f(I) \subset J$.

Define $h : J \to \mathbb{R}$ by

$$h(y) = \begin{cases} \dfrac{g(y) - g(y_0)}{y - y_0} - g'(y_0), & y \neq y_0 \\ 0, & y = y_0 \end{cases}.$$

Since $y_0 \in D(f)$, we see

$$\lim_{y \to y_0} h(y) = \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} - g'(y_0) = g'(y_0) - g'(y_0) = 0 = h(0),$$

so $y_0 \in C(h)$. Now, $x_0 \in C(f)$ and $f(x_0) = y_0 \in C(h)$, so Theorem 6.3.6 implies $x_0 \in C(h \circ f)$. In particular

$$\lim_{x \to x_0} h \circ f(x) = 0. \tag{7.1}$$

From the definition of $h \circ f$ for $x \in I$ with $f(x) \neq f(x_0)$, we can solve for

$$g \circ f(x) - g \circ f(x_0) = (h \circ f(x) + g' \circ f(x_0))(f(x) - f(x_0)). \tag{7.2}$$

Notice that (7.2) is also true when $f(x) = f(x_0)$. Divide both sides of (7.2) by $x - x_0$, and use (7.1) to obtain

$$\begin{aligned} \lim_{x \to x_0} \frac{g \circ f(x) - g \circ f(x_0)}{x - x_0} &= \lim_{x \to x_0} (h \circ f(x) + g' \circ f(x_0)) \frac{f(x) - f(x_0)}{x - x_0} \\ &= (0 + g' \circ f(x_0))f'(x_0) \\ &= g' \circ f(x_0)f'(x_0). \end{aligned}$$

□

Theorem 7.1.5. *Suppose $f : (a, b) \to (c, d)$ is continuous and invertible. If $x_0 \in D(f)$ and $f'(x_0) \neq 0$ for some $x_0 \in (a, b)$, then $f(x_0) \in D(f^{-1})$ and $\left(f^{-1}\right)'(f(x_0)) = 1/f'(x_0)$.*

Proof. Let $y_0 = f(x_0)$ and suppose y_n is any sequence in $f([a, b]) \setminus \{y_0\}$ converging to y_0 and $x_n = f^{-1}(y_n)$. By Theorem 6.5.5, f^{-1} is continuous, so

$$x_0 = f^{-1}(y_0) = \lim_{n \to \infty} f^{-1}(y_n) = \lim_{n \to \infty} x_n.$$

Therefore,

$$\lim_{n \to \infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \lim_{n \to \infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} = \frac{1}{f'(x_0)}.$$

□

Example 7.1.4. It follows easily from Theorem 7.1.2 that $f(x) = x^3$ is differentiable everywhere with $f'(x) = 3x^2$. Define $g(x) = \sqrt[3]{x}$. Then $g(x) = f^{-1}(x)$. Suppose $g(y_0) = x_0$ for some $y_0 \in \mathbb{R}$. According to Theorem 7.1.5,

$$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{3x_0^2} = \frac{1}{3(g(y_0))^2} = \frac{1}{3(\sqrt[3]{y_0})^2} = \frac{1}{3y_0^{2/3}}.$$

In the same manner as Example 7.1.4, the following corollary can be proved.

Corollary 7.1.6. *Suppose $q \in \mathbb{Q}$, $f(x) = x^q$ and D is the domain of f. Then $f'(x) = qx^{q-1}$ on the set*

$$\begin{cases} D, & \text{when } q \geq 1 \\ D \setminus \{0\}, & \text{when } q < 1 \end{cases}.$$

7.2 Derivatives and Extreme Points

As is learned in calculus, the derivative is a powerful tool for determining the behavior of functions. The following theorems form the basis for much of differential calculus. First, we state a few familiar definitions.

Definition 7.2.1. Suppose $f : D \to \mathbb{R}$ and $x_0 \in D$. f is said to have a *relative maximum at* x_0 if there is a $\delta > 0$ such that $f(x) \leq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap D$. f has a *relative minimum at* x_0 if $-f$ has a relative maximum at x_0. If f has either a relative maximum or a relative minimum at x_0, then it is said that f *has a relative extreme value at* x_0.

The *absolute maximum* of f occurs at x_0 if $f(x_0) \geq f(x)$ for all $x \in D$. The definitions of *absolute minimum* and *absolute extreme* are analogous.

Examples like $f(x) = x$ on $(0, 1)$ show that even the nicest functions need not have relative extrema. Corollary 6.5.4 shows that if D is compact, then any continuous function defined on D assumes both an absolute maximum and an absolute minimum on D.

Theorem 7.2.1. *Suppose $f : (a, b) \to \mathbb{R}$. If f has a relative extreme value at x_0 and $x_0 \in D(f)$, then $f'(x_0) = 0$.*

Proof. Suppose $f(x_0)$ is a relative maximum value of f. Then there must be a $\delta > 0$ such that $f(x) \leq f(x_0)$ whenever $x \in (x_0 - \delta, x_0 + \delta)$. Since $f'(x_0)$ exists,

$$x \in (x_0 - \delta, x_0) \implies \frac{f(x) - f(x_0)}{x - x_0} \leq 0 \implies f'(x_0) = \lim_{x \uparrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \leq 0 \tag{7.3}$$

and

$$x \in (x_0, x_0 + \delta) \implies \frac{f(x) - f(x_0)}{x - x_0} \geq 0 \implies f'(x_0) = \lim_{x \downarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \geq 0. \tag{7.4}$$

Combining (7.3) and (7.4) shows $f'(x_0) = 0$.

If $f(x_0)$ is a relative minimum value of f, apply the previous argument to $-f$. □

Theorem 7.2.1 is, of course, the basis for much of a beginning calculus course. If $f : [a, b] \to \mathbb{R}$, then the extreme values of f occur at points of the set

$$C = \{x \in (a, b) : f'(x) = 0\} \cup \{\, x \in [a, b] : f'(x) \text{ does not exist}\}.$$

The elements of C are often called the *critical points* or *critical numbers* of f on $[a, b]$. To find the maximum and minimum values of f on $[a, b]$, it suffices to find its maximum and minimum on the smaller set C, which is usually finite in most elementary calculus courses.

7.3 Differentiable Functions

Differentiation becomes most useful when a function has a derivative at each point of an interval.

Definition 7.3.1. The function f is *differentiable on an open interval* I if $I \subset D(f)$. If f is differentiable on its domain, then it is said to be *differentiable*. In this case, the function f' is called the *derivative* of f.

The fundamental theorem about differentiable functions is the Mean Value Theorem. Following is its simplest form.

Lemma 7.3.1 (Rolle's Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = 0 = f(b)$, then there is a $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. Since $[a, b]$ is compact, Corollary 6.5.4 implies the existence of $x_m, x_M \in [a, b]$ such that $f(x_m) \le f(x) \le f(x_M)$ for all $x \in [a, b]$. If $f(x_m) = f(x_M)$, then f is constant on $[a, b]$ and any $c \in (a, b)$ satisfies the lemma. Otherwise, either $f(x_m) < 0$ or $f(x_M) > 0$. If $f(x_m) < 0$, then $x_m \in (a, b)$ and Theorem 7.2.1 implies $f'(x_m) = 0$. If $f(x_M) > 0$, then $x_M \in (a, b)$ and Theorem 7.2.1 implies $f'(x_M) = 0$. □

Rolle's Theorem is used to prove the more powerful versions of the Mean Value Theorem. The following two propositions are the most common. The second is the version of the Mean Value Theorem appearing in most calculus books.

Theorem 7.3.2 (Cauchy Mean Value Theorem[4])**.** *If $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are such that f and g are continuous on $[a, b]$ and differentiable on (a, b), then there is a $c \in (a, b)$ such that*

$$g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a)).$$

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[4]Theorem 7.3.2 is also often called the *Generalized Mean Value Theorem.*

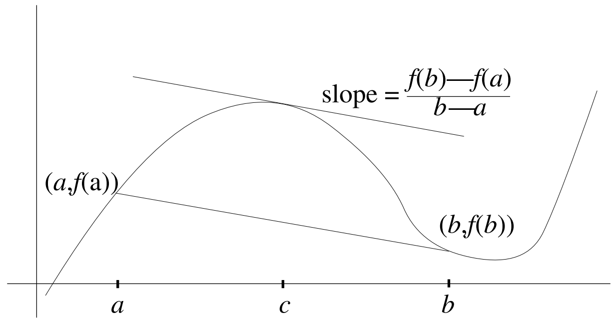

Figure 7.1: This is a "picture proof" of Corollary 7.3.3.

Proof. Let

$$h(x) = (g(b) - g(a))(f(a) - f(x)) + (g(x) - g(a))(f(b) - f(a)).$$

Then h is continuous on $[a, b]$ and differentiable on (a, b) with $h(a) = h(b) = 0$. Theorem 7.3.1 yields a $c \in (a, b)$ such that $h'(c) = 0$. Then

$$\begin{aligned} 0 = h'(c) &= -(g(b) - g(a))f'(c) + g'(c)(f(b) - f(a)) \\ &\implies g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a)). \end{aligned}$$

□

Corollary 7.3.3 (Mean Value Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there is a $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.*

Proof. Let $g(x) = x$ in Theorem 7.3.2. □

Many of the standard theorems of beginning calculus are easy consequences of the Mean Value Theorem. For example, following are the usual theorems about monotonicity.

Theorem 7.3.4. *Suppose $f : (a, b) \to \mathbb{R}$ is a differentiable function. f is increasing on (a, b) iff $f'(x) \geq 0$ for all $x \in (a, b)$. f is decreasing on (a, b) iff $f'(x) \leq 0$ for all $x \in (a, b)$.*

Proof. Only the first assertion is proved because the proof of the second is pretty much the same with all the inequalities reversed.

($\Rightarrow$) If $x, y \in (a, b)$ with $x \neq y$, then the assumption that f is increasing gives

$$\frac{f(y) - f(x)}{y - x} \geq 0 \implies f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} \geq 0.$$

($\Leftarrow$) Let $x, y \in (a, b)$ with $x < y$. According to Theorem 7.3.3, there is a $c \in (x, y)$ such that $f(y) - f(x) = f'(c)(y - x) \geq 0$. This shows $f(x) \leq f(y)$, so f is increasing on (a, b). □

Corollary 7.3.5. *Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. f is constant iff $f'(x) = 0$ for all $x \in (a, b)$.*

It follows at once from Theorem 7.1.1 that every differentiable function is continuous. But, it's not true that a *derivative* need be continuous.

Example 7.3.1. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

We claim f is differentiable everywhere, but f' is not continuous.

To see this, first note that when $x \neq 0$, the standard differentiation formulas give that $f'(x) = 2x \sin(1/x) - \cos(1/x)$. To calculate $f'(0)$, choose any $h \neq 0$. Then

$$\left| \frac{f(h)}{h} \right| = \left| \frac{h^2 \sin(1/h)}{h} \right| \leq \left| \frac{h^2}{h} \right| = |h|$$

and it easily follows from the definition of the derivative and the Squeeze Theorem (Theorem 6.1.2) that $f'(0) = 0$.

Let $x_n = 1/2\pi n$ for $n \in \mathbb{N}$. Then $x_n \to 0$ and

$$f'(x_n) = 2x_n \sin(1/x_n) - \cos(1/x_n) = -1$$

for all n. Therefore, $f'(x_n) \to -1 \neq 0 = f'(0)$, and f' is not continuous at 0.

But, derivatives do share one useful property with continuous functions; they satisfy an intermediate value property. Compare the following theorem with Corollary 6.5.7.

Theorem 7.3.6 (Darboux's Theorem)**.** *If f is differentiable on an open set containing $[a, b]$ and γ is between $f'(a)$ and $f'(b)$, then there is a $c \in [a, b]$ such that $f'(c) = \gamma$.*

Proof. If $f'(a) = f'(b)$, then $c = a$ satisfies the theorem. So, we may as well assume $f'(a) \neq f'(b)$. There is no generality lost in assuming $f'(a) < f'(b)$, for, otherwise, we just replace f with $g = -f$.

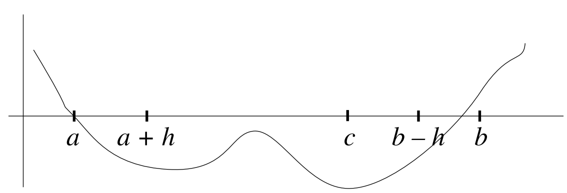

Figure 7.2: This could be the function h of Theorem 7.3.6.

Let $h(x) = f(x) - \gamma(x - \alpha)$ so that $D(f) = D(h)$ and $h'(x) = f'(x) - \gamma$. In particular, this implies $h'(a) < 0 < h'(b)$. Because of this, there must be an $\varepsilon > 0$ small enough so that

$$\frac{h(a + \varepsilon) - h(a)}{\varepsilon} < 0 \implies h(a + \varepsilon) < h(a)$$

and

$$\frac{h(b) - h(b-\varepsilon)}{\varepsilon} > 0 \implies h(b-\varepsilon) < h(b).$$

(See Figure 7.2.) In light of these two inequalities and Theorem 6.5.4, there must be a $c \in (a, b)$ such that $h(c) = \text{glb}\,\{h(x) : x \in [a, b]\}$. Now Theorem 7.2.1 gives $0 = h'(c) = f'(c) - \gamma$, and the theorem follows. □

Here's an example showing a possible use of Theorem 7.3.6.

Example 7.3.2. Let

$$f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}.$$

Theorem 7.3.6 implies f is not a derivative.

A more striking example is the following

Example 7.3.3. Define

$$f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \text{ and } g(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ -1, & x = 0 \end{cases}.$$

Since

$$f(x) - g(x) = \begin{cases} 0, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

does not have the intermediate value property, at least one of f or g is not a derivative. (Actually, neither is a derivative because $f(x) = -g(-x)$.)

7.4 Applications of the Mean Value Theorem

In the following sections, the standard notion of higher order derivatives is used. To make this precise, suppose f is defined on an interval I. The function f itself can be written $f^{(0)}$. If f is differentiable, then f' is written $f^{(1)}$. Continuing inductively, if $n \in \omega$, $f^{(n)}$ exists on I and $x_0 \in D(f^{(n)})$, then $f^{(n+1)}(x_0) = df^{(n)}(x_0)/dx$.

7.4.1 Taylor's Theorem

The motivation behind Taylor's theorem is the attempt to approximate a function f near a number a by a polynomial. The polynomial of degree 0 which does the best job is clearly $p_0(x) = f(a)$. The best polynomial of degree 1 is the tangent line to the graph of the function $p_1(x) = f(a) + f'(a)(x-a)$. Continuing in this way, we approximate f near a by the polynomial p_n of degree n such

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that $f^{(k)}(a) = p_n^{(k)}(a)$ for $k = 0, 1, \ldots, n$. A simple induction argument shows that

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{7.5}$$

This is the well-known Taylor polynomial of f at a.

Many students leave calculus with the mistaken impression that (7.5) is the important part of Taylor's theorem. But, the important part of Taylor's theorem is the fact that in many cases it is possible to determine how large n must be to achieve a desired accuracy in the approximation of f; i. e., the error term is the important part.

Theorem 7.4.1 (Taylor's Theorem)**.** *If f is a function such that $f, f', \ldots, f^{(n)}$ are continuous on $[a, b]$ and $f^{(n+1)}$ exists on (a, b), then there is a $c \in (a, b)$ such that*

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Proof. Let the constant α be defined by

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{\alpha}{(n+1)!}(b-a)^{n+1} \tag{7.6}$$

and define

$$F(x) = f(b) - \left(\sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(b-x)^k + \frac{\alpha}{(n+1)!}(b-x)^{n+1} \right).$$

From (7.6) we see that $F(a) = 0$. Direct substitution in the definition of F shows that $F(b) = 0$. From the assumptions in the statement of the theorem, it is easy to see that F is continuous on $[a, b]$ and differentiable on (a, b). An application of Rolle's Theorem yields a $c \in (a, b)$ such that

$$0 = F'(c) = -\left(\frac{f^{(n+1)}(c)}{n!}(b-c)^n - \frac{\alpha}{n!}(b-c)^n \right) \implies \alpha = f^{(n+1)}(c),$$

as desired. □

Now, suppose f is defined on an open interval I with $a, x \in I$. If f is $n+1$ times differentiable on I, then Theorem 7.4.1 implies there is a c between a and x such that

$$f(x) = p_n(x) + R_f(n, x, a),$$

where $R_n(c, x, a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ is the error in the approximation.

Example 7.4.1. Let $f(x) = \cos(x)$. Suppose we want to approximate $f(2)$ to 5 decimal places of accuracy. Since it's an easy point to work with, we'll choose $a = 0$. Then, for some $c \in (0, 2)$,

$$|R_f(n, 2, 0)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}2^{n+1} \leq \frac{2^{n+1}}{(n+1)!}. \tag{7.7}$$

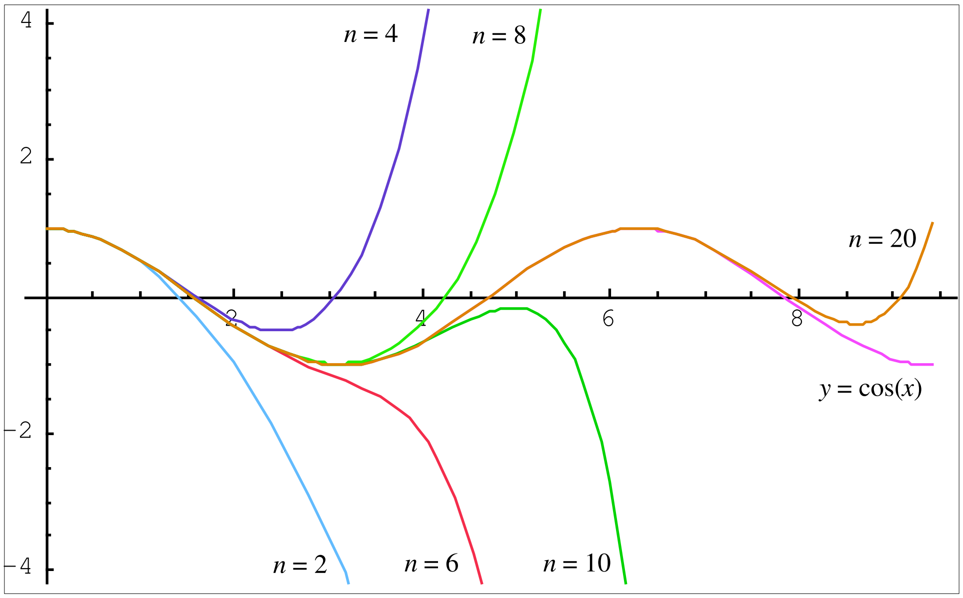

Figure 7.3: Here are several of the Taylor polynomials for the function $f(x) = \cos(x)$ graphed along with f.

A bit of experimentation with a calculator shows that $n = 12$ is the smallest n such that the right-hand side of (7.7) is less than 5×10^{-6}. After doing some arithmetic, it follows that

$$p_{12}(2) = 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + \frac{2^8}{8!} - \frac{2^{10}}{10!} + \frac{2^{12}}{12!} = -\frac{27809}{66825} \approx -0.41614.$$

is a 5 decimal place approximation to $\cos(2)$.

But, things don't always work out the way we might like. Consider the following example.

Example 7.4.2. Suppose

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

In Example 7.4.4 it is shown $f^{(n)}(0) = 0$ for all $n \geq 0$. With this function the Taylor polynomial centered at 0 gives a useless approximation.

7.4.2 L'Hôspital's Rules and Indeterminate Forms

According to Theorem 6.1.3,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

whenever $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and $\lim_{x\to a} g(x) \neq 0$. But, it is easy to find examples where both $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} f(x)/g(x)$ exists, as well as similar examples where $\lim_{x\to a} f(x)/g(x)$ fails to exist. Because of this, such a limit problem is said to be in the *indeterminate form* $0/0$. The following theorem allows us to determine many such limits.

Theorem 7.4.2 (Easy L'Hôspital's Rule)**.** *Suppose f and g are each continuous on $[a, b]$, differentiable on (a, b) and $f(b) = g(b) = 0$. If $g'(x) \neq 0$ on (a, b) and $\lim_{x\uparrow b} f'(x)/g'(x) = L$, where L could be infinite, then $\lim_{x\uparrow b} f(x)/g(x) = L$.*

Proof. Let $x \in [a, b)$, so f and g are continuous on $[x, b]$ and differentiable on (x, b). Cauchy's Mean Value Theorem, Theorem 7.3.2, implies there is a $c(x) \in (x, b)$ such

$$f'(c(x))g(x) = g'(c(x))f(x) \implies \frac{f(x)}{g(x)} = \frac{f'(c(x))}{g'(c(x))}.$$

Since $x < c(x) < b$, it follows that $\lim_{x\uparrow b} c(x) = c$. This shows that

$$L = \lim_{x\uparrow b} \frac{f'(x)}{g'(x)} = \lim_{x\uparrow b} \frac{f'(c(x))}{g'(c(x))} = \lim_{x\uparrow b} \frac{f(x)}{g(x)}.$$

□

Several things should be noted about this proof. First, there is nothing special about the left-hand limit used in the statement of the theorem. It could just as easily be written in terms of the right-hand limit. Second, if $\lim_{x\to a} f(x)/g(x)$ is not of the indeterminate form $0/0$, then applying L'Hôspital's rule will give a wrong answer. To see this, consider

$$\lim_{x\to 0} \frac{x}{x+1} = 0 \neq 1 = \lim_{x\to 0} \frac{1}{1}.$$

Corollary 7.4.3. *Suppose f and g are differentiable on (a, ∞) and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$. If $g'(x) \neq 0$ on (a, ∞) and $\lim_{x\to\infty} f'(x)/g'(x) = L$, where L could be infinite, then $\lim_{x\to\infty} f(x)/g(x) = L$.*

Proof. There is no generality lost by assuming $a > 0$. Let

$$F(x) = \begin{cases} f(1/x), & x \in (0, 1/a] \\ 0, & x = 0 \end{cases} \quad \text{and} \quad G(x) = \begin{cases} g(1/x), & x \in (0, 1/a] \\ 0, & x = 0 \end{cases}.$$

Then

$$\lim_{x\downarrow 0} F(x) = \lim_{x\to\infty} f(x) = 0 = \lim_{x\to\infty} g(x) = \lim_{x\downarrow 0} G(x),$$

so both F and G are continuous at 0. It follows that both F and G are continuous on $[0, 1/a]$ and differentiable on $(0, 1/a)$ with $G'(x) = -g'(x)/x^2 \neq 0$ on $(0, 1/a)$ and $\lim_{x\downarrow 0} F'(x)/G'(x) = \lim_{x\to\infty} f'(x)/g'(x) = L$. The rest follows from Theorem 7.4.2. □

The other standard indeterminate form arises when

$$\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x).$$

This is called an ∞/∞ indeterminate form. This is handled by the following theorem.

Theorem 7.4.4 (Hard L'Hôspital's Rule)**.** *Suppose that f and g are differentiable on (a,∞) and $g'(x) \neq 0$ on (a,∞). If*

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \cup \{-\infty, \infty\},$$

then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

Proof. First, suppose $L \in \mathbb{R}$ and let $\varepsilon > 0$. Choose $a_1 > a$ large enough so that

$$\left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon, \ \forall x > a_1.$$

Since $\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x)$, we can assume there is an $a_2 > a_1$ such that both $f(x) > 0$ and $g(x) > 0$ when $x > a_2$. Finally, choose $a_3 > a_2$ such that whenever $x > a_3$, then $f(x) > f(a_2)$ and $g(x) > g(a_2)$.

Let $x > a_3$ and apply Cauchy's Mean Value Theorem, Theorem 7.3.2, to f and g on $[a_2, x]$ to find a $c(x) \in (a_2, x)$ such that

$$\frac{f'(c(x))}{g'(c(x))} = \frac{f(x) - f(a_2)}{g(x) - g(a_2)} = \frac{f(x)\left(1 - \frac{f(a_2)}{f(x)}\right)}{g(x)\left(1 - \frac{g(a_2)}{g(x)}\right)}. \tag{7.8}$$

If

$$h(x) = \frac{1 - \frac{g(a_2)}{g(x)}}{1 - \frac{f(a_2)}{f(x)}},$$

then (7.8) implies

$$\frac{f(x)}{g(x)} = \frac{f'(c(x))}{g'(c(x))} h(x).$$

Since $\lim_{x\to\infty} h(x) = 1$, there is an $a_4 > a_3$ such that whenever $x > a_4$, then $|h(x) - 1| < \varepsilon$. If $x > a_4$, then

$$\begin{aligned}
\left| \frac{f(x)}{g(x)} - L \right| &= \left| \frac{f'(c(x))}{g'(c(x))} h(x) - L \right| \\
&= \left| \frac{f'(c(x))}{g'(c(x))} h(x) - Lh(x) + Lh(x) - L \right| \\
&\leq \left| \frac{f'(c(x))}{g'(c(x))} - L \right| |h(x)| + |L||h(x) - 1| \\
&< \varepsilon(1 + \varepsilon) + |L|\varepsilon = (1 + |L| + \varepsilon)\varepsilon
\end{aligned}$$

can be made arbitrarily small through a proper choice of ε. Therefore $\lim_{x\to\infty} f(x)/g(x) = L$.

The case when $L = \infty$ is done similarly by first choosing a $B > 0$ and adjusting (7.8) so that $f'(x)/g'(x) > B$ when $x > a_1$. A similar adjustment is necessary when $L = -\infty$. □

There is a companion corollary to Theorem 7.4.4 which is proved in the same way as Corollary 7.4.3.

Corollary 7.4.5. *Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b) with $g'(x) \neq 0$ on (a, b). If*

$$\lim_{x\downarrow a} f(x) = \lim_{x\downarrow a} g(x) = \infty \quad \textit{and} \quad \lim_{x\downarrow a} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \cup \{-\infty, \infty\},$$

then

$$\lim_{x\downarrow a} \frac{f(x)}{g(x)} = L.$$

Example 7.4.3. If $\alpha > 0$, then $\lim_{x\to\infty} \ln x/x^\alpha$ is of the indeterminate form ∞/∞. Taking derivatives of the numerator and denominator yields

$$\lim_{x\to\infty} \frac{1/x}{\alpha x^{\alpha-1}} = \lim_{x\to\infty} \frac{1}{\alpha x^\alpha} = 0.$$

Theorem 7.4.4 now implies $\lim_{x\to\infty} \ln x/x^\alpha = 0$, and therefore $\ln x$ increases more slowly than any positive power of x.

Example 7.4.4. Let f be as in Example 7.4.2. It is clear $f^{(n)}(x)$ exists whenever $n \in \omega$ and $x \neq 0$. We claim $f^{(n)}(0) = 0$. To see this, we first prove that

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^n} = 0, \ \forall n \in \mathbb{Z}. \tag{7.9}$$

When $n \leq 0$, (7.9) is obvious. So, suppose (7.9) is true whenever $m \leq n$ for some $n \in \omega$. Making the substitution $u = 1/x$, we see

$$\lim_{x\downarrow 0} \frac{e^{-1/x^2}}{x^{n+1}} = \lim_{u\to\infty} \frac{u^{n+1}}{e^{u^2}}. \tag{7.10}$$

Since

$$\lim_{u\to\infty} \frac{(n+1)u^n}{2ue^{u^2}} = \lim_{u\to\infty} \frac{(n+1)u^{n-1}}{2e^{u^2}} = \frac{n+1}{2} \lim_{x\downarrow 0} \frac{e^{-1/x^2}}{x^{n-1}} = 0$$

by the inductive hypothesis, Theorem 7.4.4 gives (7.10) in the case of the right-hand limiit. The left-hand limit is handled similarly. Finally, (7.9) follows by induction.

When $x \neq 0$, $f^{(n)}(x)$ is easily seen to be of the form $p_n(1/x)e^{-1/x^2}$, where p_n is a polynomial. Induction and repeated applications of (7.9) establish that $f^{(n)}(0) = 0$ for $n \in \omega$.

7.5 Problems

Problem 60. If f is defined on an open set containing x_0, the *symmetric derivative* of f at x_0 is defined as

$$f^s(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h}.$$

Prove that if $f'(x)$ exists, then so does $f^s(x)$. Is the converse true?

Problem 61. Finish Example 7.4.2.

Problem 62. Let G be an open set and $f \in D(G)$. If there is an $a \in G$ such that $\lim_{x \to a} f'(x)$ exists, then $\lim_{x \to a} f'(x) = f'(a)$.

Problem 63. Prove or give a counter example: If $f \in D((a, b))$ such that f' is bounded, then there is an $F \in C([a, b])$ such that $f = F$ on (a, b).

Problem 64. Suppose f is continuous on $[a, b]$ and f'' exists on (a, b). If there is an $x_0 \in (a, b)$ such that the line segment between $(a, f(a))$ and $(b, f(b))$ contains the point $(x_0, f(x_0))$, then there is a $c \in (a, b)$ such that $f''(c) = 0$.

Problem 65. Prove that

$$\left| \sin x - \left(x - \frac{x^3}{6} + \frac{x^5}{120} \right) \right| < \frac{1}{5040}$$

when $|x| \le 1$.

Problem 66. Prove or give a counter example: If f is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R} \setminus \{0\}$ with $\lim_{x \to 0} f'(x) = L$, then f is differentiable on $\mathbb{R}$.

Problem 67. Let f be defined on a neighborhood of x.

(a) If $f''(x)$ exists, then

$$\lim_{h \to 0} \frac{f(x - h) - 2f(x) + f(x + h)}{h^2} = f''(x).$$

(b) Find a function f where this limit exists, but $f''(x)$ does not exist.