INROADS

Real Analysis Exchange
Vol. 36(2), 2010/2011, pp. 463–470

Jacek Hejduk, Faculty of Mathematics and Computer Science, University of Łódź, Banacha 22, PL-90-238 Łódź, Poland.
email: hejduk@math.uni.lodz.pl

Renata Wiertelak, Faculty of Mathematics and Computer Science, University of Łódź, Banacha 22, PL-90-238 Łódź, Poland.
email: wiertelak@math.uni.lodz.pl

CONTINUOUS FUNCTIONS IN $\mathcal{I}(J)$–DENSITY TOPOLOGIES

Abstract

This paper contains the properties of continuous functions equipped with the $\mathcal{I}(J)$–density topology or natural topology in the domain or the range.

Let $\mathbb{R}$ be the set of reals and $\mathbb{N}$ stand for the set of natural numbers. Let $\mathcal{I}$ be the σ–ideal of first category sets in $\mathbb{R}$, $\mathcal{S}$ be the σ–algebra of sets having the Baire property in $\mathbb{R}$, and $\mathcal{T}_{nat}$ be the natural topology in $\mathbb{R}$.

According to paper [4], we shall say that 0 is a density point with respect to category of a set $A \in \mathcal{S}$ if the sequence $\{f_n\}_{n\in\mathbb{N}} = \{\chi_{nA\cap[-1,1]}\}_{n\in\mathbb{N}}$ converges with respect to the σ–ideal $\mathcal{I}$ to the characteristic function $\chi_{[-1,1]}$. It means that every subsequence of the sequence $\{f_n\}_{n\in\mathbb{N}}$ contains a subsequence converging to the function $\chi_{[-1,1]}$ everywhere except for a set of the first category. For $J = [a,b]$ let us put

$$\begin{aligned} s(J) &= \frac{1}{2}(a+b), \\ h(A,J)(x) &= \chi_{\frac{2}{|J|}(A-s(J))\cap[-1,1]}(x), \end{aligned}$$

where $A + z = \{a + z : a \in A\}$, $\alpha A = \{\alpha a : a \in A\}$ for $z, \alpha \in \mathbb{R}, A \subset \mathbb{R}$. By $J = \{J_n\}_{n\in\mathbb{N}}$ we shall denote a non–degenerate **sequence of intervals tending to zero**, that means

$$\lim_{n\to\infty} s(J_n) = 0 \qquad \wedge \qquad \lim_{n\to\infty} |J_n| = 0.$$

Mathematical Reviews subject classification: Primary: 26A15; Secondary: 58C05
Key words: real functions, continuity, approximate continuity
Received by the editors April 13, 2010
Communicated by: Udayan B. Darji

If a sequence of intervals $J = \{J_n\}_{n\in\mathbb{N}}$ is tending to zero and $J_n \subset [0,\infty)$ ($J_n \subset (-\infty, 0]$) for $n \in \mathbb{N}$, then we say that the sequence J is **tending to zero from the right (left) side**.

The point 0 is called an $\mathcal{I}(J)$–**density point** of a set $A \in \mathcal{S}$ if

$$h(A, J_n)(x) \xrightarrow[n\to\infty]{\mathcal{I}} \chi_{[-1,1]}(x).$$

It means that

$$\underset{\{n_k\}_{k\in\mathbb{N}}}{\forall} \quad \underset{\{n_{k_m}\}_{m\in\mathbb{N}}}{\exists} \quad \underset{\Theta\in\mathcal{I}}{\exists} \quad \underset{x\notin\Theta}{\forall} \quad h(A, J_{n_{k_m}})(x) \xrightarrow[m\to\infty]{} \chi_{[-1,1]}(x).$$

It is obvious that 0 is an $\mathcal{I}(J)$–density point of a set $A \in \mathcal{S}$ if and only if

$$\underset{\{n_k\}_{k\in\mathbb{N}}}{\forall} \quad \underset{\{n_{k_m}\}_{m\in\mathbb{N}}}{\exists} \quad \limsup_{m\to\infty} \left([-1,1] \setminus (A - s(J_{n_{k_m}})) \frac{2}{|J_{n_{k_m}}|} \right) \in \mathcal{I}.$$

We shall say that a point $x_0 \in \mathbb{R}$ is an $\mathcal{I}(J)$–density point of a set $A \in \mathcal{S}$ if and only if 0 is an $\mathcal{I}(J)$–density point of the set $A - x_0$.

A point $x_0 \in \mathbb{R}$ is an $\mathcal{I}(J)$–dispersion point of a set $A \in \mathcal{S}$ if and only if x_0 is an $\mathcal{I}(J)$–density point of the complementary set A'.

It is easy to see that if $J_n = \left[-\frac{1}{n}, \frac{1}{n}\right]$ for $n \in \mathbb{N}$, then x_0 is an $\mathcal{I}(J)$–density point of a set $A \in \mathcal{S}$ if and only if x_0 is an $\mathcal{I}$–density point of A (see [4]). When $J_n = \left[-\frac{1}{s_n}, \frac{1}{s_n}\right]$ for $n \in \mathbb{N}$, where $\langle s \rangle = \{s_n\}_{n\in\mathbb{N}}$ is an unbounded and nondecreasing sequence of positive real numbers, then the notion of the $\mathcal{I}(J)$–density point of a set $A \in \mathcal{S}$ is equivalent to the notion of the $\langle s \rangle$–density point of A (see [2]).

If $A \in \mathcal{S}$, then we denote

$$\Phi_{\mathcal{I}(J)}(A) = \{x \in \mathbb{R} \colon x \text{ is an } \mathcal{I}(J)\text{–density point of } A\}.$$

Theorem 1. *(cf [5]) If $J = \{J_n\}_{n\in\mathbb{N}}$ is a sequence of intervals tending to zero, then the operator $\Phi_{\mathcal{I}(J)} \colon \mathcal{S} \to \mathcal{S}$ is the lower density operator on $(\mathbb{R}, \mathcal{S}, \mathcal{I})$ and the family*

$$\mathcal{T}_{\mathcal{I}(J)} = \{A \in \mathcal{S} \colon A \subset \Phi_{\mathcal{I}(J)}(A)\}.$$

is a topology on $\mathbb{R}$, which will be called an $\mathcal{I}(J)$–density topology, such that $\mathcal{T}_{nat} \subset \mathcal{T}_{\mathcal{I}(J)}$.

If $J = \{J_n\}_{n\in\mathbb{N}}$, where $J_n = [a_n, b_n]$, is a sequence of intervals tending to zero and $m \in \mathbb{R} \setminus \{0\}$, then $mJ = \{mJ_n\}_{n\in\mathbb{N}}$, where $mJ_n = [ma_n, mb_n]$ for $m > 0$ and $mJ_n = [mb_n, ma_n]$ for $m < 0$, is the sequence of intervals tending to zero as well.

From the definition of an $\mathcal{I}(J)$–density point and an $\mathcal{I}(J)$–density topology it is easy to conclude the following property.

Property 2. *If $J = \{J_n\}_{n\in\mathbb{N}}$ is a sequence of intervals tending to zero, then for every set $A \in \mathcal{S}$ the following properties holds:*

(i) $\underset{x\in\mathbb{R}}{\forall}\ \underset{a\in\mathbb{R}}{\forall}\ x \in \Phi_{\mathcal{I}(J)}(A) \Leftrightarrow (x+a) \in \Phi_{\mathcal{I}(J)}(A+a);$

(ii) $\underset{x\in\mathbb{R}}{\forall}\ \underset{m\neq 0}{\forall}\ x \in \Phi_{\mathcal{I}(J)}(A) \Leftrightarrow mx \in \Phi_{\mathcal{I}(mJ)}(mA);$

(iii) $\underset{a\in\mathbb{R}}{\forall}\ A \in \mathcal{T}_{\mathcal{I}(J)} \Leftrightarrow (A+a) \in \mathcal{T}_{\mathcal{I}(J)};$

(iv) $\underset{m\neq 0}{\forall}\ A \in \mathcal{T}_{\mathcal{I}(J)} \Leftrightarrow mA \in \mathcal{T}_{\mathcal{I}(mJ)}.$

Also the next property is a consequence of an $\mathcal{I}(J)$–density point.

Property 3. *(cf [5]) If $J = \{J_n\}_{n\in\mathbb{N}}$ is a sequence of intervals tending to zero, then the point 0 is an $\mathcal{I}(J)$–density point of the set*

$$A_k = \{0\} \cup \bigcup_{n\geq k} int(J_n),$$

for every $k \in \mathbb{N}$. Moreover $A_k \in \mathcal{T}_{\mathcal{I}(J)}$.

Likewise in the case of an $\mathcal{I}$–density topology (see [1], [7]) the following property of an $\mathcal{I}(J)$–density topology holds.

Property 4. *A set A is compact with respect to an $\mathcal{I}(J)$–density topology if and only if A is finite.*

Much more interesting properties of $\mathcal{I}(J)$–density topologies can be found in the papers [5], [6]. We recall those of them which are necessary in further considerations.

Theorem 5. *(cf [6]) If $J = \{J_n\}_{n\in\mathbb{N}}$ is a sequence of intervals tending to zero from the right (left) side, then every set $[a,b)$ ($(a,b]$), for $a,b \in \mathbb{R}$ and $a < b$, belongs to the topology $\mathcal{T}_{\mathcal{I}(J)}$ whereas every set $(a,b]$ ($[a,b)$) is not the member of the $\mathcal{I}(J)$–density topology.*

Theorem 6. *(cf [5]) Let $J = \{J_n\}_{n\in\mathbb{N}}$, where $J_n = [a_n, b_n]$ for $n \in \mathbb{N}$, be a sequence of intervals tending to zero and*

$$K_n^i = \left[a_n + \frac{i-1}{l_0}(b_n - a_n), a_n + \frac{i}{l_0}(b_n - a_n)\right],$$

for $n \in \mathbb{N}$, $l_0 \in \mathbb{N}$, $i \in \{1, \ldots, l_0\}$. Then the family $\left\{K_n^i\right\}_{i\in\{1,\ldots,l_0\}, n\in\mathbb{N}}$ ordered in the sequence

$$K = \left\{K_1^1, K_1^2, \ldots K_1^{l_0}, K_2^1, K_2^2, \ldots, K_2^{l_0}, \ldots\right\}$$

is tending to zero and $\mathcal{T}_{\mathcal{I}(J)} = \mathcal{T}_{\mathcal{I}(K)}$.

Let $J = \{J_n\}_{n\in\mathbb{N}}$ be a sequence of intervals tending to zero. Then we obtain four families of continuous functions defined as follows:

$$\begin{aligned}\mathcal{C}_{nat,nat} &= \{f\colon (\mathbb{R}, \mathcal{T}_{nat}) \to (\mathbb{R}, \mathcal{T}_{nat})\}\\ \mathcal{C}_{nat,\mathcal{I}(J)} &= \{f\colon (\mathbb{R}, \mathcal{T}_{nat}) \to (\mathbb{R}, \mathcal{T}_{\mathcal{I}(J)})\}\\ \mathcal{C}_{\mathcal{I}(J),nat} &= \{f\colon (\mathbb{R}, \mathcal{T}_{\mathcal{I}(J)}) \to (\mathbb{R}, \mathcal{T}_{nat})\}\\ \mathcal{C}_{\mathcal{I}(J),\mathcal{I}(J)} &= \{f\colon (\mathbb{R}, \mathcal{T}_{\mathcal{I}(J)}) \to (\mathbb{R}, \mathcal{T}_{\mathcal{I}(J)})\}.\end{aligned}$$

Functions of the family $\mathcal{C}_{\mathcal{I}(J),nat}$ will be called $\mathcal{I}(J)$–approximately continuous functions and functions of $\mathcal{C}_{\mathcal{I}(J),\mathcal{I}(J)}$ will be called $\mathcal{I}(J)$–continuous.

Property 7. *The family $\mathcal{C}_{nat,\mathcal{I}(J)}$ consists of constant functions.*

PROOF. Let $f \in \mathcal{C}_{nat,\mathcal{I}(J)}$ and $a, b \in \mathbb{R}$ such that $a < b$. Then $f([a, b])$ is nonempty, compact and connected set with respect to the topology $\mathcal{T}_{\mathcal{I}(J)}$. By Property 4 this compact set is finite. Moreover the set $f([a, b])$ is connected and as a result $f(a) = f(b)$. For that reason the function f is constant and the proof is completed. □

The next property is an easy consequence of the inclusion $\mathcal{T}_{nat} \subset \mathcal{T}_{\mathcal{I}(J)}$.

Property 8. *For every sequence of intervals J the following inclusions holds:*

(i) $\mathcal{C}_{nat,\mathcal{I}(J)} \subset \mathcal{C}_{nat,nat} \subset \mathcal{C}_{\mathcal{I}(J),nat}$

(ii) $\mathcal{C}_{nat,\mathcal{I}(J)} \subset \mathcal{C}_{\mathcal{I}(J),\mathcal{I}(J)} \subset \mathcal{C}_{\mathcal{I}(J),nat}$.

Moreover inclusions $\mathcal{C}_{nat,\mathcal{I}(J)} \subset \mathcal{C}_{nat,nat}$ and $\mathcal{C}_{nat,\mathcal{I}(J)} \subset \mathcal{C}_{\mathcal{I}(J),\mathcal{I}(J)}$ are proper. Indeed, the identical function is the member of $\mathcal{C}_{nat,nat}$ and $\mathcal{C}_{\mathcal{I}(J),\mathcal{I}(J)}$ but not $\mathcal{C}_{nat,\mathcal{I}(J)}$.

Property 9. *If $J = \{J_n\}_{n\in\mathbb{N}}$ is a sequence of intervals tending to zero from the right or left side, then:*

(i) $\mathcal{C}_{nat,nat} \setminus \mathcal{C}_{\mathcal{I}(J),\mathcal{I}(J)} \neq \emptyset$;

(ii) $\mathcal{C}_{\mathcal{I}(J),\mathcal{I}(J)} \setminus \mathcal{C}_{nat,nat} \neq \emptyset$.

PROOF. Let us suppose that the sequence J is tending to zero from the right side. To show the first inclusion we consider the function $f(x) = -x^2$. Obviously $f \in \mathcal{C}_{nat,nat}$. This and inclusion $\mathcal{C}_{nat,nat} \subset \mathcal{C}_{\mathcal{I}(J),nat}$ imply that $f \in \mathcal{C}_{\mathcal{I}(J),nat}$. Further the set $A = [-1, 1) \in \mathcal{T}_{\mathcal{I}(J)}$ (by Theorem 5), whereas $f^{-1}(A) = [-1, 1] \notin \mathcal{T}_{\mathcal{I}(J)}$. It means that $f \notin \mathcal{C}_{\mathcal{I}(J),\mathcal{I}(J)}$.

To prove the second inclusion we define the function

$$h(x) = x - k \qquad \text{for } x \in [k, k+1), k \in \mathbb{Z}.$$

It is easy to see that for every set $B \subset \mathbb{R}$ holds

$$h^{-1}(B) = \bigcup_{k \in \mathbb{Z}} \big((B \cap [0,1)) + k\big).$$

Thus for every set $B \in \mathcal{T}_{\mathcal{I}(J)}$ we have that $h^{-1}(B) \in \mathcal{T}_{\mathcal{I}(J)}$ (by Theorem 5 and Property 2). Therefore $h \in \mathcal{C}_{\mathcal{I}(J),\mathcal{I}(J)}$. Moreover inclusion $\mathcal{C}_{\mathcal{I}(J),\mathcal{I}(J)} \subset \mathcal{C}_{\mathcal{I}(J),nat}$ implies that $f \in \mathcal{C}_{\mathcal{I}(J),nat}$. Simultaneously

$$h^{-1}\left(\left(-1, \frac{1}{2}\right)\right) = \bigcup_{k \in \mathbb{Z}} \left[k, k + \frac{1}{2}\right) \notin \mathcal{T}_{nat}.$$

Hence $h \notin \mathcal{C}_{nat,nat}$ and inclusion (ii) is proper.

If J is tending to zero from the left side, then we consider the sets $A = (-1, 1]$, $B = \left(\frac{1}{2}, 2\right)$ and the functions $f(x) = x^2$, $h(x) = x - k$ for $x \in (k, k+1]$, $k \in \mathbb{Z}$. □

An immediate consequence of this proof is the following corollary.

Corollary 10. *Let $J = \{J_n\}_{n \in \mathbb{N}}$ be a sequence of intervals tending to zero from the right or left side. Then the inclusions*

(i) $\mathcal{C}_{\mathcal{I}(J),\mathcal{I}(J)} \subset \mathcal{C}_{\mathcal{I}(J),nat}$

(ii) $\mathcal{C}_{nat,nat} \subset \mathcal{C}_{\mathcal{I}(J),nat}$

are proper.

Let $J = \{J_n\}_{n \in \mathbb{N}}$ and $K = \{K_n\}_{n \in \mathbb{N}}$ be sequences of intervals. Then the sequence ordered in an arbitrary fashion containing all intervals of the sequences J and K, denoted by $J \cup K$, is called **the union of sequences** J and K.

Remark 11. *If J and K are sequences tending to zero, then the sequence $J \cup K$ is also tending to zero. It is evident from the definition of an $\mathcal{I}(J)$–density point that an $\mathcal{I}(J \cup K)$–density topology is independent of the ordering of intervals in the sequence $J \cup K$.*

Property 12. *If $J = \{J_n\}_{n \in \mathbb{N}}$ and $K = \{K_n\}_{n \in \mathbb{N}}$ are sequences of intervals tending to zero, then*

$$\mathcal{T}_{\mathcal{I}(J \cup K)} = \mathcal{T}_{\mathcal{I}(J)} \cap \mathcal{T}_{\mathcal{I}(K)}.$$

Properties 12 and 7 yields to the following property.

Property 13. *Let $J = \{J_n\}_{n\in\mathbb{N}}$ and $K = \{K_n\}_{n\in\mathbb{N}}$ be sequences of intervals tending to zero. Then*

(i) $\mathcal{C}_{\mathcal{I}(J),nat} \cap \mathcal{C}_{\mathcal{I}(K),nat} = \mathcal{C}_{\mathcal{I}(J\cup K),nat}$*;*

(ii) $\mathcal{C}_{nat,\mathcal{I}(J)} \cap \mathcal{C}_{nat,\mathcal{I}(K)} = \mathcal{C}_{nat,\mathcal{I}(J\cup K)}$*;*

(iii) $\mathcal{C}_{\mathcal{I}(J),\mathcal{I}(J)} \cap \mathcal{C}_{\mathcal{I}(K),\mathcal{I}(K)} \subset \mathcal{C}_{\mathcal{I}(J\cup K),\mathcal{I}(J\cup K)}$*.*

Moreover there are sequences J and K for which

$$\mathcal{C}_{\mathcal{I}(J\cup K),\mathcal{I}(J\cup K)} \setminus \left(\mathcal{C}_{\mathcal{I}(J),\mathcal{I}(J)} \cap \mathcal{C}_{\mathcal{I}(K),\mathcal{I}(K)}\right) \neq \emptyset.$$

PROOF. The conditions (i), (ii) and the inclusion (iii) are evident. We prove that $\mathcal{C}_{\mathcal{I}(J\cup K),\mathcal{I}(J\cup K)} \setminus \left(\mathcal{C}_{\mathcal{I}(J),\mathcal{I}(J)} \cap \mathcal{C}_{\mathcal{I}(K),\mathcal{I}(K)}\right)$ is non–empty. Putting $J_n = \left[0, \frac{1}{n}\right]$, $K_n = \left[-\frac{1}{n}, 0\right]$ for $n \in \mathbb{N}$ we obtain the sequences $J = \{J_n\}_{n\in\mathbb{N}}$ and $K = \{K_n\}_{n\in\mathbb{N}}$ of intervals tending to zero. By Theorem 6 the topology $\mathcal{T}_{\mathcal{I}(J\cup K)}$ is the $\mathcal{I}$–density topology. Hence the function $f(x) = -x$ belongs to the family $\mathcal{C}_{\mathcal{I}(J\cup K),\mathcal{I}(J\cup K)}$. Since the sequence J is tending to zero from the right side, thus $[0, 1) \in \mathcal{T}_{\mathcal{I}(J)}$, whereas $f^{-1}([0, 1)) = (-1, 0] \notin \mathcal{T}_{\mathcal{I}(J)}$ (by Theorem 5). It implies that $f \notin \mathcal{C}_{\mathcal{I}(J),\mathcal{I}(J)}$. Using similar argument we can show that $f \notin \mathcal{C}_{\mathcal{I}(K),\mathcal{I}(K)}$. □

Now we will investigate $\mathcal{I}(J)$–continuity of a function $f(x) = ax$.

Theorem 14. *A function $f(x) = ax$ is $\mathcal{I}(J)$–continuous for every sequence of intervals $J = \{J_n\}_{n\in\mathbb{N}}$ tending to zero if and only if $a \in \{0, 1\}$.*

PROOF. Sufficiency is obvious because the constant function and the identity function are $\mathcal{I}(J)$–continuous for every sequence J tending to zero.

Necessity. If $a < 0$, then for every sequence J tending to zero from the right side the function f is not $\mathcal{I}(J)$–continuous. Indeed, if we consider the set $A = [0, 1)$, then $A \in \mathcal{T}_{\mathcal{I}(J)}$ and $f^{-1}(A) = (-a, 0] \notin \mathcal{T}_{\mathcal{I}(J)}$ by Theorem 5. Thus the function f is not $\mathcal{I}(J)$–continuous.

If $a > 0$ and $a \neq 1$, then we define $J_n = [b^{2n+1}, b^{2n}]$, where $b = min\{a, a^{-1}\}$, and put $J = \{J_n\}_{n\in\mathbb{N}}$. The sequence J is tending to zero and by Property 3 the set

$$A = \{0\} \cup \bigcup_{n\in\mathbb{N}} \left(b^{2n+1}, b^{2n}\right)$$

belongs to the topology $\mathcal{T}_{\mathcal{I}(J)}$, whereas

$$f^{-1}(A) = \{0\} \cup \bigcup_{n\in\mathbb{N}} \left(a^{-1}b^{2n+1}, a^{-1}b^{2n}\right) \subset \mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} J_n.$$

It shows that 0 is the $\mathcal{I}(J)$–density point of the set $\big(\mathbb{R}\setminus f^{-1}(A)\big)\supset\bigcup_{n\in\mathbb{N}}J_n$. It implies that 0 is not the $\mathcal{I}(J)$–density point of the set $f^{-1}(A)$. Therefore $f^{-1}(A)\notin\mathcal{T}_{\mathcal{I}(J)}$. It follows that f is not the $\mathcal{I}(J)$–continuous function. □

The following corollary is an immediate consequence of the last proof.

Corollary 15. *For an arbitrary number $a\in\mathbb{R}\setminus\{0,1\}$ there exists a sequence of intervals $J=\{J_n\}_{n\in\mathbb{N}}$ tending to zero and a set A such that $A\in\mathcal{T}_{\mathcal{I}(J)}$ and $a^{-1}A\notin\mathcal{T}_{\mathcal{I}(J)}$.*

Theorem 16. *For any sequence of intervals $J=\{J_n\}_{n\in\mathbb{N}}$ tending to zero and any number $a\neq 0$ the function $f(x)=ax$ is $\mathcal{I}(J)$–continuous if and only if $\mathcal{T}_{\mathcal{I}(J)}\subset\mathcal{T}_{\mathcal{I}(aJ)}$.*

Proof. Necessity. Let $A\in\mathcal{T}_{\mathcal{I}(J)}$. By the $\mathcal{I}(J)$–density continuity we have that

$$f^{-1}(A)=a^{-1}A\in\mathcal{T}_{\mathcal{I}(J)}.$$

Theorem 2 implies that $A\in\mathcal{T}_{\mathcal{I}(aJ)}$. Hence $\mathcal{T}_{\mathcal{I}(J)}\subset\mathcal{T}_{\mathcal{I}(aJ)}$.

Sufficiency. Let $A\in\mathcal{T}_{\mathcal{I}(J)}$ and $a\neq 0$. Then $A\in\mathcal{T}_{\mathcal{I}(aJ)}$ and by Property 2 (iv) we have that $a^{-1}A\in\mathcal{T}_{\mathcal{I}(J)}$. Since $f^{-1}(A)=a^{-1}A$, therefore $f^{-1}(A)\in\mathcal{T}_{\mathcal{I}(J)}$. It follows that the function f is $\mathcal{I}(J)$–continuous. □

References

[1] K. Ciesielski, L. Larson, and K. Ostaszewski, *$\mathcal{I}$-Density Continous Functions*, Mem. Amer. Math. Soc. 107, **515**, 1994.

[2] J. Hejdukand G. Horbaczewska, *On $\mathcal{I}$-density topologies with respect to a fixed sequence*, Reports on Real Analysis, Conference at Rowy, (2003), 78–85.

[3] G. Horbaczewska, *The family of $\mathcal{I}$-density type topologies*, Comment. Math. Univ. Carolinae, **46(4)** (2005), 735–745.

[4] W. Poreda, E. Wagner-Bojakowska, and W. Wilczyński, *A category analogue of the density topology*, Fund. Math., **125** (1985), 167-173.

[5] R. Wiertelak, *A generalization of density topology with respect to category*, Real Anal. Exchange, **32(1)** (2006/2007), 273–286.

[6] R. Wiertelak, *About $\mathcal{I}(J)$–approximately continuous functions*, Period. Math. Hungar., (submitted).

[7] W. Wilczyński, *A generalization of density topology*, Real Anal. Exchange, **8(1)** (1982-83), 16–20.