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ANOTHER PROOF THAT L^p-BOUNDED POINTWISE CONVERGENCE IMPLIES WEAK CONVERGENCE

Abstract

This note gives another proof of the known fact that L^p-bounded pointwise convergence implies weak convergence in L^p, $p > 1$. The proof is based on Banach and Saks's theorem. The same method applies to convergence in measure.

One of the results that connect pointwise convergence of Lebesgue measurable functions (real analysis) and weak convergence in the space L^p (functional analysis) is the following

Theorem 1. *Let $f^i \in L^p(\Omega)$, $1 < p < +\infty$, $\Omega \subseteq \mathbb{R}^N$. Suppose that the sequence $\{f^i\}$ is bounded in the norm. If f^i converges pointwise a.e. to f, then f^i converges weakly to f in $L^p(\Omega)$. In other words, L^p-bounded pointwise convergence implies weak convergence.*

A proof of this theorem can be found, for instance, in [2, Theorem 13.44]. That proof uses classical theorems of real analysis: Fatou's lemma, the absolute continuity of the Lebesgue integral, and Egorov's theorem.

The objective of this note is to prove Theorem 1 in another way. The proof proposed here uses two theorems from functional analysis.

Theorem 2 (F. Riesz). *Let $\{f^j\} \subseteq L^p$, $p > 1$, be bounded in L^p. Then it contains a subsequence $\{f^{j_k}\}$ that converges weakly in L^p. See [3, Theorem 2.18].*

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Theorem 3 (Banach and Saks)**.** *Let $\left\{f^k\right\} \subseteq L^p$, $p > 1$, converge weakly to f^0 in L^p. There exists a subsequence $\left\{f^{k_l}\right\}$ such that $\frac{1}{n}\sum_{l=1}^{n} f^{k_l}$ converges strongly to f^0 in L^p.*

This theorem [1, § 4] is a precursor of S. Mazur's theorem on strongly convergent sequences built out of weakly convergent ones. Its original proof involves elementary inequalities and Riesz's theorem stated above. A short proof for $p = 2$ is given in [4].

We shall need three well-known propositions.

Proposition 1. *Let $s^n \in L^p(\Omega)$, $p \geq 1$, $\Omega \subseteq \mathbb{R}^N$. If $\{s^n\}$ tends strongly to s in $L^p(\Omega)$, then there exists a subsequence $\{s^{n_q}\}$ that converges pointwise a.e. to s.*

Proposition 2. *Given a sequence $\left\{\alpha^i\right\} \subseteq \mathbb{R}$, if each subsequence $\left\{\alpha^{i_j}\right\}$ contains a subsequence $\left\{\alpha^{i_{j_k}}\right\}$ that converges to α, then $\alpha^i \longrightarrow \alpha$.*

Proposition 3. *Given a sequence $\left\{\alpha^l\right\} \subseteq \mathbb{R}$, if $\alpha^l \longrightarrow \alpha$, then $\frac{1}{n}\sum_{l=1}^{n} \alpha^l \longrightarrow \alpha$.*

Proof of Theorem 1. Take any subsequence $\left\{f^{i_j}\right\}$. Since it is bounded in $L^p(\Omega)$, it is possible to extract a subsequence $\left\{f^{i_{j_k}}\right\}$ that converges weakly to some $f^0 \in L^p(\Omega)$. To finish the proof, it is enough to show that $f^0 = f$ a.e.: put $\alpha^i := \left\langle f^i, \lambda\right\rangle$ and $\alpha := \langle f, \lambda\rangle$ where λ is an arbitrary element of $L^p(\Omega)^*$, and use Proposition 2.

For convenience, the elements of the subsequence $\left\{f^{i_{j_k}}\right\}$ will be denoted by f^k. According to this notation, $\left\{f^k\right\}$ tends weakly to f^0 in $L^p(\Omega)$. By Banach and Saks's theorem there is a subsequence $\left\{f^{k_l}\right\}$ such that

$$\frac{1}{n}\sum_{l=1}^{n} f^{k_l} \longrightarrow f^0 \qquad \text{strongly in } L^p(\Omega).$$

According to Proposition 1 there exists a subsequence $\{n_q\}$ for which

$$\frac{1}{n_q}\sum_{l=1}^{n_q} f^{k_l}(x) \longrightarrow f^0(x) \qquad \text{for a.e. } x \in \Omega.$$

On the other hand, by assumption and using Proposition 3

$$\frac{1}{n_q}\sum_{l=1}^{n_q} f^{k_l}(x) \longrightarrow f(x) \qquad \text{for a.e. } x \in \Omega,$$

which proves that $f^0 = f$ a.e. □

In the same way, we can prove

Theorem 4. *Let $f^i \in L^p(\Omega)$, $1 < p < +\infty$, $\Omega \subseteq \mathbb{R}^N$. Suppose that the sequence $\{f^i\}$ is bounded in the norm. If f^i converges in measure to f, then f^i converges weakly to f in $L^p(\Omega)$: L^p-bounded convergence in measure implies weak convergence.*

A slight modification of the proof for this case consists in extracting, at the very beginning, a pointwise convergent subsequence out of $\{f^{i_j}\}$ and only then choosing a weakly convergent subsequence.

References

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