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CHARACTERIZATIONS OF SOME SUBCLASSES OF THE FIRST CLASS OF BAIRE

Abstract

In the paper [3] the authors have examined functions of the Baire class 1, where the domain and the range were metric spaces. The $\varepsilon - \delta$ characterization of such functions has been proved. In this note we examine, if replacing of the condition from [3] by it's stronger version can lead us to the characterization of some subclass of B_1 on the interval $[0, 1]$.

1 Definition of the class B_A. Basic properties

In 2000 year Lee, Tang and Zhao published the following theorem:

Theorem ([3]) Suppose that $f : X \to \mathbf{R}$ is a real valued function on a complete separable metric space X. Then the following statements are equivalent:

(1) For every $\varepsilon > 0$ there exists a positive function δ on X such that

$$|f(x_1) - f(x_2)| < \varepsilon$$

whenever

$$d_X(x_1, x_2) < \min(\delta(x_1), \delta(x_2)).$$

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(2) The function f is of Baire class one.

Following Atok, Tang and Zhao ([4]) we call the positive function δ in (1) an ε-gauge of f.

We restrict our considerations to the case of real functions defined on $[0,1]$ interval. The question is: If we use only gauges δ meeting some extra conditions, are we supposed to obtain the class smaller then B_1? The answer is positive. To express it more precisely consider the following definition :

Definition 1 Let A be an arbitrary family of real valued functions defined on the interval $[0,1]$, $f : [0,1] \to \mathbf{R}$. We say that $f \in B_A$, iff for every $\varepsilon > 0$ there exists $\delta : [0,1] \to (0,\infty)$ such that $\delta \in A$ and for every $x_1, x_2 \in [0,1]$ the following implication holds

$$|x_1 - x_2| < \min(\delta(x_1), \delta(x_2)) \quad \Longrightarrow \quad |f(x_1) - f(x_2)| < \varepsilon.$$

In the sequel let $B1$, C, lsc, usc, D, app, ls-app, $B1^*$ denote respectively classes of Baire 1, continuous, lower semicontinuous, upper semicontinuous, Darboux continuous, approximately continuous, lower semi approximately continuous functions and Baire* one functions defined on $[0,1]$.

(Function $f : [0,1] \to \mathbf{R}$ is *lower semi approximately continuous* iff for each $a \in \mathbf{R}$ the set $f^{-1}((a,\infty))$ is open with respect to the density topology on $[0,1]$.

Function $f : [0,1] \to \mathbf{R}$ belongs to $B1^*$ iff for every closed set $F \subset [0,1]$ there exists the interval (a,b) such that $(a,b) \cap F \neq \emptyset$ and the function f restricted to $(a,b) \cap F$ is continuous.)

Proposition 1 *The operator* $A \to B_A$ *has the following properties:*

1. *If* $A_1 \subset A_2 \subset \mathbf{R}^{[0,1]}$ *then* $B_{A_1} \subset B_{A_2} \subset B1$,
2. *For every family* $A \subset \mathbf{R}^{[0,1]}$ *the family* B_A *is closed under uniform convergence,*
3. *If the family* $A \subset \mathbf{R}^{[0,1]}$ *satisfies the following condition:*

$$\min(f,g) \in A \quad \text{for} \quad \text{every} \quad f,g \in A,$$

then the family B_A *forms a linear subspace of the space* $\mathbf{R}^{[0,1]}$.

Proof.

1. The first inclusion follows directly from the definition of the family B_A. The second inclusion follows from the Lee, Tang and Zao theorem.

2. Let $f_n \in B_A$ for $n \in \mathbf{N}$, and the sequence (f_n) converges uniformly to the function f. Take $\varepsilon > 0$. Let $n \in \mathbf{N}$ be large enough to get
$$|f(x) - f_n(x)| < \frac{\varepsilon}{3}$$
for every $x \in [0,1]$. Let $\delta \in A$ be an $\frac{\varepsilon}{3}$-gauge of the function f_n. Take arbitrary $a, b \in [0,1]$ such that $|a-b| < \min(\delta(a), \delta(b))$. Then
$$|f(a) - f(b)| \leq |f(a) - f_n(a)| + |f_n(a) - f_n(b)| + |f_n(b) - f(b)| < \varepsilon.$$
So the function δ is an ε-gauge of f. Hence $f \in B_A$.

3. Let $f, g \in B_A$, let $\alpha \in \mathbf{R}$, $\varepsilon > 0$. Let $\delta \in A$ be an $(\varepsilon \cdot (\max(|\alpha|, 1))^{-1})$-gauge of f. Then δ is an ε-gauge of αf. Let δ_1 be the $(\frac{\varepsilon}{2})$-gauge of f and δ_2 be the $(\frac{\varepsilon}{2})$-gauge of g. Then the function $\min(\delta_1, \delta_2)$ is an ε-gauge of $f+g$. □

2 Results

Theorem 1 $B_{const} = B_C = B_{lsc} = C$.

Proof. We have $B_{const} \subset B_C \subset B_{\text{lsc}}$ because $const \subset C \subset \text{lsc}$.

Let $f \in B_{\text{lsc}}$, $\varepsilon > 0$. Let $\delta \in \text{lsc}$ be an ε-gauge of f. As the function δ is positive and lower semicontinuous, it has the positive lower bound η on $[0,1]$. Hence, for every $a, b \in [0,1]$, the condition $|a-b| < \eta$ implies $|f(a) - f(b)| < \varepsilon$. Therefore the function f is uniformly continuous on $[0,1]$.

Let $f \in C$. Then f is uniformly continuous. For every $\varepsilon > 0$ there exists $\eta > 0$ such that $|f(a) - f(b)| < \varepsilon$ if $|a-b| < \eta$. Defining $\delta(x) = \eta$ for $x \in [0,1]$, we obtain a constatnt ε-gauge of f. □

Let x_0 be a right hand side accumulation point of the set $E \subset \mathbf{R}$. By $\text{Lim}_{x \to x_0+} f(x)$ we shall denote the set of all limits of sequences of the form $f(x_n)$, where (x_n) is the sequence of points belonging to $E \cap (x_0, \infty)$ converging to x_0. The set $\text{Lim}_{x \to x_0+} f(x)$ is always closed. The definition of $\text{Lim}_{x \to x_0-} f(x)$ is analogous.

Let us recall the characterization of Darboux continuous functions among the Baire one functions:

Theorem ([1], chap. 2, th. 1.1) *Let $f : [0,1] \to \mathbf{R}$ be of Baire class one. Then the following conditions are equivalent*

(1) f has the Darboux property;

(2) $f(x_0) \in Lim_{x\to x_0-} f(x) \cap Lim_{x\to x_0+} f(x)$ for every $x_0 \in [0,1]$.

(In case when $x_0 \in \{0,1\}$ the second condition has its unilateral version.)

Theorem 2 $B_D \subset D$.

Proof. Let $f \in B_D$. Of course $f \in B_1$. Suppose that $f \notin D$. From the last theorem we have existence of $x_0 \in [0,1]$, such that $f(x_0) \notin \mathrm{Lim}_{x\to x_0-} f(x) \cap \mathrm{Lim}_{x\to x_0+} f(x)$ For instance let the distance between $f(x_0)$ and $\mathrm{Lim}_{x\to x_0-} f(x)$ equals 2ε for some positive number ε. Let δ be an ε-gauge of f. We shall show, that δ cannot be Darboux continuous. Let $(x_n)_{n\in\mathbf{N}}$ be an arbitrary choosen sequence of points belonging to $[0, x_0)$ converging to x_0. There exists N_1 such that $|x_n - x_0| < \delta(x_0)$ for $n > N_1$. And there exists N_2 such that $|f(x_n) - f(x_0)| > \varepsilon$ for $n > N_2$.

Therefore for $n > \max(N_1, N_2)$ we have $|x_n - x_0| \geq \delta(x_n)$. Hence $\lim_{x\to x_0-} \delta(x) = 0$. But $\delta(x_0) > 0$, so δ is not Darboux continuous. □

Problem 1 *Does the opposite inclusion hold : $B1 \cap D \ \subset \ B_D$?*

Theorem 3 $B_{ls\text{-}app} \subset app$.

Proof. Let $f \in B_{\text{ls-app}}$, $\varepsilon > 0$, $x_0 \in [0,1]$. We shall demonstrate, that x_0 is a density point of the set $Z(x_0, \varepsilon) = f^{-1}\Big((f(x_0) - \varepsilon, f(x_0) + \varepsilon)\Big)$. Let δ be an approximately lower semicontinuous ε-gauge of f. Then x_0 is the density point of the set

$$T(x_0) = \delta^{(-1)}\Big(\big(\frac{\delta(x_0)}{2}, \infty\big)\Big)$$

Let x be an arbitrary point of $T(x_0)$ such that $|x - x_0| < \frac{\delta(x_0)}{2}$. Then $|x - x_0| < \min(\delta(x), \delta(x_0))$, so $|f(x) - f(x_0)| < \varepsilon$. Hence $x \in Z(x_0, \varepsilon)$. Therefore

$$T(x_0) \cap \left(x_0 - \frac{\delta(x_0)}{2}, x_0 + \frac{\delta(x_0)}{2}\right) \subset Z(x_0, \varepsilon).$$

But x_0 is a density point of $T(x_0) \cap \left(x_0 - \frac{\delta(x_0)}{2}, x_0 + \frac{\delta(x_0)}{2}\right)$, hence it is also the density point of $Z(x_0, \varepsilon)$. □

Let us recall the notion of *oscillation index* of function. Let $f : [0,1] \to \mathbf{R}$ and $\varepsilon > 0$. For each $A \subset [0,1]$ let

$$P_{\varepsilon,f}(A) = \Big\{x \in A \ : \ \mathrm{osc}(f,x,A) \geq \varepsilon\Big\}.$$

Let us define the transfinite sequence of sets $(F^{\alpha}_{f,\varepsilon})_{\alpha<\omega_1}$ in the following way:

$$F^{\alpha}_{f,\varepsilon} \quad = \quad \begin{cases} [0,1] & \text{for } \ \alpha = 0 \\ P_{\varepsilon,f}(F^{\gamma}_{f,\varepsilon}) & \text{for } \ \alpha = \gamma + 1 \\ \bigcap_{\gamma<\alpha} F^{\gamma}_{f,\varepsilon} & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

If there exist $\alpha < \omega_1$ such that $F^{\alpha}_{f,\varepsilon} = \emptyset$ then let $\beta(f,\varepsilon) = \min\{\alpha : F^{\alpha}_{f,\varepsilon} = \emptyset\}$. In case that for every $\alpha < \omega_1$ $F^{\alpha}_{f,\varepsilon} \neq \emptyset$ let $\beta(f,\varepsilon) = \omega_1$. Finally let $\beta(f) = \sup_{\varepsilon>0} \beta(f,\varepsilon)$.

Let us recall the following
Theorem ([2]) *Let $f : [0,1] \to \mathbf{R}$. Then*

(1) $\beta(f) = 1$ iff f is continuous,

(2) $\beta(f) < \omega_1$ iff $f \in B1$.

The next theorem shows the connection between the oscillation index of the function f and the oscillation index of its gauge:

Theorem 4 *Let $f : [0,1] \to \mathbf{R}$, $\beta(f) = \alpha < \omega_1$. Then for every $\varepsilon > 0$ there exists an ε-gauge δ of f such that $\delta \in usc \cap B1^*$ and $\beta(\delta) \leq \alpha$.*

Proof. Let $\varepsilon > 0$. We shall construct a gauge δ using the sequence $\{F^{\gamma}_{f,\varepsilon}\}_{\gamma\leq\alpha}$ defined in the definition of the oscillation index. Notice that

$$[0,1] \ = \ \bigcup_{\gamma<\alpha} (F^{\gamma}_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon}).$$

Suppose that $\gamma < \alpha$ and the value $\delta(x)$ has been already defined for $x \in \bigcup_{\xi<\gamma}(F^{\xi}_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}) = [0,1] \setminus F^{\gamma}_{f,\varepsilon}$. Now we define the function δ on the set $F^{\gamma}_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon}$.

Let $t \in F^{\gamma}_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon}$. According to the definition of the sequence $\{F^{\gamma}_{f,\varepsilon}\}_{\gamma \leq \alpha}$ the oscillation of the function f restricted to the set $F^{\gamma}_{f,\varepsilon}$ at the point t is less than ε. So there exists an open neighbourhood V_t of t, such that $\operatorname{diam}(f(V_t \cap F^{\gamma}_{f,\varepsilon})) < \varepsilon$. Let us define the function $\delta^t(x) = \frac{1}{2} d(x, [0,1] \setminus V_t)$, where $d(x, A)$ stands for the distance between the point x and the set A. The function δ^t satisfies the Lipschitz condition with the constant 1.

Let

$$\delta(x) = \sup_{t \in F^{\gamma}_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon}} \delta^t(x)$$

for $x \in F^{\gamma}_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon}$. Observe that the function δ is strictly positive. It is also continuous on the set $F^{\gamma}_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon}$ as an upper bound of an equi-continuous family of functions.

In this way we have defined the gauge δ on the whole interval $[0,1]$.

Now we shall examine the properties of δ.

(1) Let $x \in [0,1], x \in F^{\gamma}_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon}, \gamma < \alpha$. Then $(x - \delta(x), x + \delta(x)) \cap F^{\gamma+1}_{f,\varepsilon} = \emptyset$. In fact, by the definition of δ, there exists $t \in F^{\gamma}_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon}$, such that $\delta^t(x) > \frac{3}{4}\delta(x)$. Therefore there exists a set V_t such that $\operatorname{diam}(f(V_t \cap F^{\gamma}_{f,\varepsilon})) < \varepsilon$ and $d(x, [0,1] \setminus V_t) = 2\delta^t(x) > \frac{3}{2}\delta(x)$. Hence $(x - \delta(x), x + \delta(x)) \subset V_t$ and $V_t \cap F^{\gamma+1}_{f,\varepsilon} = \emptyset$.

(2) If two points $x, y \in [0,1]$ fullfill the condition $|x - y| < \min(\delta(x), \delta(y))$ then for some ordinal γ we have $x, y \in F^{\gamma}_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon}$ and $|f(x) - f(y)| < \varepsilon$. Hence δ is an ε-gauge of f.

Proof. Let $x \in F^{\gamma}_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon}$ and $y \in F^{\xi}_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}$ for $\gamma, \xi < \alpha$. Assume that $|x - y| < \min(\delta(x), \delta(y))$ and suppose that $\gamma < \xi$. Since

$$y \in (x - \delta(x), x + \delta(x)) \subset [0,1] \setminus F^{\gamma+1}_{f,\varepsilon},$$

we get $y \notin F^{\gamma+1}_{f,\varepsilon}$, which contradicts $F^{\xi}_{f,\varepsilon} \subset F^{\gamma+1}_{f,\varepsilon}$. So $\gamma = \xi$.

Again from the definition of δ, there exists $t \in F^{\gamma}_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon}$, such that $\delta^t(x) > \frac{3}{4}\delta(x)$. So $d(x, [0,1] \setminus V_t) > \frac{3}{2}\delta(x) > |x - y|$. Hence $x, y \in V_t$. From the definition of V_t it follows that $|f(x) - f(y)| < \varepsilon$.

(3) The function δ restricted to the set $F^{\gamma}_{f,\varepsilon}$ is continuous on the set $F^{\gamma}_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon}$.

Proof. The function δ is defined on the set $F^{\gamma}_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon}$ as an upper bound of the family of functions that fullfill the Lipschitz condition with common constant 1. Moreover the set $F^{\gamma}_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon}$ is open with respect to $F^{\gamma}_{f,\varepsilon}$.

(4) $\beta(\delta) \leq \alpha$.

Proof. For a given $\eta > 0$ let us consider the sequence of sets $\{F^{\gamma}_{\delta,\eta}\}_{\gamma<\omega_1}$. We shall show by the transfinite induction that for every $\gamma < \omega_1$ we have

$$F^{\gamma}_{\delta,\eta} \subset F^{\gamma}_{f,\varepsilon}.$$

The inclusion is obvious for $\gamma = 0$ because $[0, 1] \subset [0, 1]$. Suppose that for $\gamma < \omega_1$ the above inclusion holds. From *(3)* it follows that the oscillation of the function δ is equal to 0 in each point of the set $F^{\gamma}_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon}$ Hence $F^{\gamma+1}_{\delta,\eta} \subset F^{\gamma+1}_{f,\varepsilon}$. So $\beta(\delta, \eta) \leq \beta(f, \varepsilon) \leq \alpha$, and, as the number η is arbitrary, we have $\beta(\delta) \leq \alpha$.

(5) The function δ is upper semicontinuous.
Consider $x_0 \in [0, 1]$ and a sequence $(x_n)_{n\in\mathbf{N}}$ convergent to x_0 such that $\lim_{n\to\infty} \delta(x_n) = g$. We shall show that $g \leq \delta(x_0)$. There exists the ordinal $\gamma < \alpha$, such that $x_0 \in F^{\gamma}_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon}$. As $x_0 \notin F^{\gamma+1}_{f,\varepsilon}$ and the set $F^{\gamma+1}_{f,\varepsilon}$ is closed we can assume that no term of the sequence (x_n) belongs to $F^{\gamma+1}_{f,\varepsilon}$. There are two possibilities:

1° almost every term of the sequence (x_n) belongs to $F^{\gamma}_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon}$;

2° infinitely many terms of that sequence belongs to $[0, 1] \setminus F^{\gamma}_{f,\varepsilon}$.

In the first case we have $g = \lim_{n\to\infty} \delta(x_n) = \delta(x_0)$ as the function δ is continuous on $F^{\gamma}_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon}$.

In the second case we can assume that all terms of (x_n) belong to $[0, 1]\setminus F^{\gamma}_{f,\varepsilon}$. For a given n in virtue of *(1)* we obtain

$$x_0 \notin (x_n - \delta(x_n), x_n + \delta(x_n)),$$

hence

$$|x_0 - x_n| \geq \delta(x_n) > 0,$$

and

$$g = \lim_{n\to\infty} \delta(x_n) = 0 < \delta(x_0).$$

As a result δ is upper semicontinuous.

(6) The function δ belongs to $B1^*$.
Let $F \subset [0, 1]$ be an arbitrary nonempty closed set. We shall show existence of the interval (a, b) such that $(a, b) \cap F \neq \emptyset$ and $\delta|_F$ is continuous on $(a, b) \cap F$. Let γ be the least ordinal such that $F \subset F^{\gamma}_{f,\varepsilon}$ but $F \not\subset F^{\gamma+1}_{f,\varepsilon}$. Let $x_0 \in F \cap (F^{\gamma}_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon})$. From *(1)* and *(3)*, the function δ is continuous on the set $(x_0 - \delta(x_0), x_0 + \delta(x_0)) \cap F^{\gamma}_{f,\varepsilon}$, and hence continuous on the set $(x_0 - \delta(x_0), x_0 + \delta(x_0)) \cap F$. As a consequence $\delta \in B1^*$. □

Remark 1 The last result is similar (but not comparable) to the main result of Atok, Tang and Zhao ([4], Theorem 2). Nevertheless we decided to demonstrate our theorem because the proof is completely different.

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