

Monika Lindner, Center of Mathematics and Physics, Technical University of Łódź, Al. Politechniki 11 90-924 Łódź, Poland. email: mlindner@p.lodz.pl

Sebastian Lindner, Faculty of Mathematics and Computer Science, University of Łódź, 90-238 Łódź, Banacha 22, Poland. email: lindner@math.uni.lodz.pl

## CHARACTERIZATIONS OF SOME SUBCLASSES OF THE FIRST CLASS OF BAIRE

### Abstract

In the paper [3] the authors have examined functions of the Baire class 1, where the domain and the range were metric spaces. The  $\varepsilon - \delta$  characterization of such functions has been proved. In this note we examine, if replacing of the condition from [3] by it's stronger version can lead us to the characterization of some subclass of  $B_1$  on the interval  $[0, 1]$ .

### 1 Definition of the class $B_A$ . Basic properties

In 2000 year Lee, Tang and Zhao published the following theorem:

**Theorem ([3])** Suppose that  $f : X \rightarrow \mathbf{R}$  is a real valued function on a complete separable metric space  $X$ . Then the following statements are equivalent:

- (1) For every  $\varepsilon > 0$  there exists a positive function  $\delta$  on  $X$  such that

$$|f(x_1) - f(x_2)| < \varepsilon$$

whenever

$$d_X(x_1, x_2) < \min(\delta(x_1), \delta(x_2)).$$

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(2) The function  $f$  is of Baire class one.

Following Atok, Tang and Zhao ([4]) we call the positive function  $\delta$  in (1) an  $\varepsilon$ -gauge of  $f$ .

We restrict our considerations to the case of real functions defined on  $[0, 1]$  interval. The question is: If we use only gauges  $\delta$  meeting some extra conditions, are we supposed to obtain the class smaller than  $B_1$ ? The answer is positive. To express it more precisely consider the following definition :

**Definition 1** Let  $A$  be an arbitrary family of real valued functions defined on the interval  $[0, 1]$ ,  $f : [0, 1] \rightarrow \mathbf{R}$ . We say that  $f \in B_A$ , iff for every  $\varepsilon > 0$  there exists  $\delta : [0, 1] \rightarrow (0, \infty)$  such that  $\delta \in A$  and for every  $x_1, x_2 \in [0, 1]$  the following implication holds

$$|x_1 - x_2| < \min(\delta(x_1), \delta(x_2)) \implies |f(x_1) - f(x_2)| < \varepsilon.$$

In the sequel let  $B_1$ ,  $C$ , lsc, usc,  $D$ , app, ls-app,  $B_1^*$  denote respectively classes of Baire 1, continuous, lower semicontinuous, upper semicontinuous, Darboux continuous, approximately continuous, lower semi approximately continuous functions and Baire\* one functions defined on  $[0, 1]$ .

(Function  $f : [0, 1] \rightarrow \mathbf{R}$  is *lower semi approximately continuous* iff for each  $a \in \mathbf{R}$  the set  $f^{-1}((a, \infty))$  is open with respect to the density topology on  $[0, 1]$ .)

Function  $f : [0, 1] \rightarrow \mathbf{R}$  belongs to  $B_1^*$  iff for every closed set  $F \subset [0, 1]$  there exists the interval  $(a, b)$  such that  $(a, b) \cap F \neq \emptyset$  and the function  $f$  restricted to  $(a, b) \cap F$  is continuous.)

**Proposition 1** *The operator  $A \rightarrow B_A$  has the following properties:*

1. *If  $A_1 \subset A_2 \subset \mathbf{R}^{[0,1]}$  then  $B_{A_1} \subset B_{A_2} \subset B_1$ ,*
2. *For every family  $A \subset \mathbf{R}^{[0,1]}$  the family  $B_A$  is closed under uniform convergence,*
3. *If the family  $A \subset \mathbf{R}^{[0,1]}$  satisfies the following condition:*

$$\min(f, g) \in A \quad \text{for every } f, g \in A,$$

*then the family  $B_A$  forms a linear subspace of the space  $\mathbf{R}^{[0,1]}$ .*

Proof.

1. The first inclusion follows directly from the definition of the family  $B_A$ . The second inclusion follows from the Lee, Tang and Zao theorem.
2. Let  $f_n \in B_A$  for  $n \in \mathbf{N}$ , and the sequence  $(f_n)$  converges uniformly to the function  $f$ . Take  $\varepsilon > 0$ . Let  $n \in \mathbf{N}$  be large enough to get

$$|f(x) - f_n(x)| < \frac{\varepsilon}{3}$$

for every  $x \in [0, 1]$ . Let  $\delta \in A$  be an  $\frac{\varepsilon}{3}$ -gauge of the function  $f_n$ . Take arbitrary  $a, b \in [0, 1]$  such that  $|a - b| < \min(\delta(a), \delta(b))$ . Then

$$|f(a) - f(b)| \leq |f(a) - f_n(a)| + |f_n(a) - f_n(b)| + |f_n(b) - f(b)| < \varepsilon.$$

So the function  $\delta$  is an  $\varepsilon$ -gauge of  $f$ . Hence  $f \in B_A$ .

3. Let  $f, g \in B_A$ , let  $\alpha \in \mathbf{R}$ ,  $\varepsilon > 0$ . Let  $\delta \in A$  be an  $(\varepsilon \cdot (\max(|\alpha|, 1))^{-1})$ -gauge of  $f$ . Then  $\delta$  is an  $\varepsilon$ -gauge of  $\alpha f$ . Let  $\delta_1$  be the  $(\frac{\varepsilon}{2})$ -gauge of  $f$  and  $\delta_2$  be the  $(\frac{\varepsilon}{2})$ -gauge of  $g$ . Then the function  $\min(\delta_1, \delta_2)$  is an  $\varepsilon$ -gauge of  $f + g$ .  $\square$

## 2 Results

**Theorem 1**  $B_{const} = B_C = B_{lsc} = C$ .

Proof. We have  $B_{const} \subset B_C \subset B_{lsc}$  because  $const \subset C \subset lsc$ .

Let  $f \in B_{lsc}$ ,  $\varepsilon > 0$ . Let  $\delta \in lsc$  be an  $\varepsilon$ -gauge of  $f$ . As the function  $\delta$  is positive and lower semicontinuous, it has the positive lower bound  $\eta$  on  $[0, 1]$ . Hence, for every  $a, b \in [0, 1]$ , the condition  $|a - b| < \eta$  implies  $|f(a) - f(b)| < \varepsilon$ . Therefore the function  $f$  is uniformly continuous on  $[0, 1]$ .

Let  $f \in C$ . Then  $f$  is uniformly continuous. For every  $\varepsilon > 0$  there exists  $\eta > 0$  such that  $|f(a) - f(b)| < \varepsilon$  if  $|a - b| < \eta$ . Defining  $\delta(x) = \eta$  for  $x \in [0, 1]$ , we obtain a constant  $\varepsilon$ -gauge of  $f$ .  $\square$

Let  $x_0$  be a right hand side accumulation point of the set  $E \subset \mathbf{R}$ . By  $\text{Lim}_{x \rightarrow x_0+} f(x)$  we shall denote the set of all limits of sequences of the form  $f(x_n)$ , where  $(x_n)$  is the sequence of points belonging to  $E \cap (x_0, \infty)$  converging to  $x_0$ . The set  $\text{Lim}_{x \rightarrow x_0+} f(x)$  is always closed. The definition of  $\text{Lim}_{x \rightarrow x_0-} f(x)$  is analogous.

Let us recall the characterization of Darboux continuous functions among the Baire one functions:

**Theorem** ([1], chap. 2, th. 1.1) *Let  $f : [0, 1] \rightarrow \mathbf{R}$  be of Baire class one. Then the following conditions are equivalent*

(1)  *$f$  has the Darboux property;*

(2)  *$f(x_0) \in \text{Lim}_{x \rightarrow x_0^-} f(x) \cap \text{Lim}_{x \rightarrow x_0^+} f(x)$  for every  $x_0 \in [0, 1]$ .*

*(In case when  $x_0 \in \{0, 1\}$  the second condition has its unilateral version.)*

**Theorem 2**  $B_D \subset D$ .

Proof. Let  $f \in B_D$ . Of course  $f \in B_1$ . Suppose that  $f \notin D$ . From the last theorem we have existence of  $x_0 \in [0, 1]$ , such that  $f(x_0) \notin \text{Lim}_{x \rightarrow x_0^-} f(x) \cap \text{Lim}_{x \rightarrow x_0^+} f(x)$ . For instance let the distance between  $f(x_0)$  and  $\text{Lim}_{x \rightarrow x_0^-} f(x)$  equals  $2\varepsilon$  for some positive number  $\varepsilon$ . Let  $\delta$  be an  $\varepsilon$ -gauge of  $f$ . We shall show, that  $\delta$  cannot be Darboux continuous. Let  $(x_n)_{n \in \mathbf{N}}$  be an arbitrary chosen sequence of points belonging to  $[0, x_0)$  converging to  $x_0$ . There exists  $N_1$  such that  $|x_n - x_0| < \delta(x_0)$  for  $n > N_1$ . And there exists  $N_2$  such that  $|f(x_n) - f(x_0)| > \varepsilon$  for  $n > N_2$ .

Therefore for  $n > \max(N_1, N_2)$  we have  $|x_n - x_0| \geq \delta(x_n)$ . Hence  $\lim_{x \rightarrow x_0^-} \delta(x) = 0$ . But  $\delta(x_0) > 0$ , so  $\delta$  is not Darboux continuous.  $\square$

**Problem 1** *Does the opposite inclusion hold :  $B_1 \cap D \subset B_D$ ?*

**Theorem 3**  $B_{ls-app} \subset app$ .

Proof. Let  $f \in B_{ls-app}$ ,  $\varepsilon > 0$ ,  $x_0 \in [0, 1]$ . We shall demonstrate, that  $x_0$  is a density point of the set  $Z(x_0, \varepsilon) = f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon))$ . Let  $\delta$  be an approximately lower semicontinuous  $\varepsilon$ -gauge of  $f$ . Then  $x_0$  is the density point of the set

$$T(x_0) = \delta^{(-1)}\left(\left(\frac{\delta(x_0)}{2}, \infty\right)\right)$$

Let  $x$  be an arbitrary point of  $T(x_0)$  such that  $|x - x_0| < \frac{\delta(x_0)}{2}$ . Then  $|x - x_0| < \min(\delta(x), \delta(x_0))$ , so  $|f(x) - f(x_0)| < \varepsilon$ . Hence  $x \in Z(x_0, \varepsilon)$ . Therefore

$$T(x_0) \cap \left(x_0 - \frac{\delta(x_0)}{2}, x_0 + \frac{\delta(x_0)}{2}\right) \subset Z(x_0, \varepsilon).$$

But  $x_0$  is a density point of  $T(x_0) \cap \left(x_0 - \frac{\delta(x_0)}{2}, x_0 + \frac{\delta(x_0)}{2}\right)$ , hence it is also the density point of  $Z(x_0, \varepsilon)$ .  $\square$

Let us recall the notion of *oscillation index* of function. Let  $f : [0, 1] \rightarrow \mathbf{R}$  and  $\varepsilon > 0$ . For each  $A \subset [0, 1]$  let

$$P_{\varepsilon, f}(A) = \left\{ x \in A : \text{osc}(f, x, A) \geq \varepsilon \right\}.$$

Let us define the transfinite sequence of sets  $(F_{f, \varepsilon}^\alpha)_{\alpha < \omega_1}$  in the following way:

$$F_{f, \varepsilon}^\alpha = \begin{cases} [0, 1] & \text{for } \alpha = 0 \\ P_{\varepsilon, f}(F_{f, \varepsilon}^\gamma) & \text{for } \alpha = \gamma + 1 \\ \bigcap_{\gamma < \alpha} F_{f, \varepsilon}^\gamma & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

If there exist  $\alpha < \omega_1$  such that  $F_{f, \varepsilon}^\alpha = \emptyset$  then let  $\beta(f, \varepsilon) = \min\{\alpha : F_{f, \varepsilon}^\alpha = \emptyset\}$ . In case that for every  $\alpha < \omega_1$   $F_{f, \varepsilon}^\alpha \neq \emptyset$  let  $\beta(f, \varepsilon) = \omega_1$ . Finally let  $\beta(f) = \sup_{\varepsilon > 0} \beta(f, \varepsilon)$ .

Let us recall the following

**Theorem ([2])** *Let  $f : [0, 1] \rightarrow \mathbf{R}$ . Then*

- (1)  $\beta(f) = 1$  iff  $f$  is continuous,
- (2)  $\beta(f) < \omega_1$  iff  $f \in B_1$ .

The next theorem shows the connection between the oscillation index of the function  $f$  and the oscillation index of its gauge:

**Theorem 4** *Let  $f : [0, 1] \rightarrow \mathbf{R}$ ,  $\beta(f) = \alpha < \omega_1$ . Then for every  $\varepsilon > 0$  there exists an  $\varepsilon$ -gauge  $\delta$  of  $f$  such that  $\delta \in \text{usc} \cap B_1^*$  and  $\beta(\delta) \leq \alpha$ .*

**Proof.** Let  $\varepsilon > 0$ . We shall construct a gauge  $\delta$  using the sequence  $\{F_{f, \varepsilon}^\gamma\}_{\gamma \leq \alpha}$  defined in the definition of the oscillation index. Notice that

$$[0, 1] = \bigcup_{\gamma < \alpha} (F_{f, \varepsilon}^\gamma \setminus F_{f, \varepsilon}^{\gamma+1}).$$

Suppose that  $\gamma < \alpha$  and the value  $\delta(x)$  has been already defined for  $x \in \bigcup_{\xi < \gamma} (F_{f, \varepsilon}^\xi \setminus F_{f, \varepsilon}^{\xi+1}) = [0, 1] \setminus F_{f, \varepsilon}^\gamma$ . Now we define the function  $\delta$  on the set  $F_{f, \varepsilon}^\gamma \setminus F_{f, \varepsilon}^{\gamma+1}$ .

Let  $t \in F_{f,\varepsilon}^\gamma \setminus F_{f,\varepsilon}^{\gamma+1}$ . According to the definition of the sequence  $\{F_{f,\varepsilon}^\gamma\}_{\gamma \leq \alpha}$  the oscillation of the function  $f$  restricted to the set  $F_{f,\varepsilon}^\gamma$  at the point  $t$  is less than  $\varepsilon$ . So there exists an open neighbourhood  $V_t$  of  $t$ , such that  $\text{diam}(f(V_t \cap F_{f,\varepsilon}^\gamma)) < \varepsilon$ . Let us define the function  $\delta^t(x) = \frac{1}{2}d(x, [0, 1] \setminus V_t)$ , where  $d(x, A)$  stands for the distance between the point  $x$  and the set  $A$ . The function  $\delta^t$  satisfies the Lipschitz condition with the constant 1.

Let

$$\delta(x) = \sup_{t \in F_{f,\varepsilon}^\gamma \setminus F_{f,\varepsilon}^{\gamma+1}} \delta^t(x)$$

for  $x \in F_{f,\varepsilon}^\gamma \setminus F_{f,\varepsilon}^{\gamma+1}$ . Observe that the function  $\delta$  is strictly positive. It is also continuous on the set  $F_{f,\varepsilon}^\gamma \setminus F_{f,\varepsilon}^{\gamma+1}$  as an upper bound of an equi-continuous family of functions.

In this way we have defined the gauge  $\delta$  on the whole interval  $[0, 1]$ .

Now we shall examine the properties of  $\delta$ .

(1) Let  $x \in [0, 1]$ ,  $x \in F_{f,\varepsilon}^\gamma \setminus F_{f,\varepsilon}^{\gamma+1}$ ,  $\gamma < \alpha$ . Then  $(x - \delta(x), x + \delta(x)) \cap F_{f,\varepsilon}^{\gamma+1} = \emptyset$ .

In fact, by the definition of  $\delta$ , there exists  $t \in F_{f,\varepsilon}^\gamma \setminus F_{f,\varepsilon}^{\gamma+1}$ , such that  $\delta^t(x) > \frac{3}{4}\delta(x)$ . Therefore there exists a set  $V_t$  such that  $\text{diam}(f(V_t \cap F_{f,\varepsilon}^\gamma)) < \varepsilon$  and  $d(x, [0, 1] \setminus V_t) = 2\delta^t(x) > \frac{3}{2}\delta(x)$ . Hence  $(x - \delta(x), x + \delta(x)) \subset V_t$  and  $V_t \cap F_{f,\varepsilon}^{\gamma+1} = \emptyset$ .

(2) If two points  $x, y \in [0, 1]$  fulfill the condition  $|x - y| < \min(\delta(x), \delta(y))$  then for some ordinal  $\gamma$  we have  $x, y \in F_{f,\varepsilon}^\gamma \setminus F_{f,\varepsilon}^{\gamma+1}$  and  $|f(x) - f(y)| < \varepsilon$ . Hence  $\delta$  is an  $\varepsilon$ -gauge of  $f$ .

Proof. Let  $x \in F_{f,\varepsilon}^\gamma \setminus F_{f,\varepsilon}^{\gamma+1}$  and  $y \in F_{f,\varepsilon}^\xi \setminus F_{f,\varepsilon}^{\xi+1}$  for  $\gamma, \xi < \alpha$ . Assume that  $|x - y| < \min(\delta(x), \delta(y))$  and suppose that  $\gamma < \xi$ . Since

$$y \in (x - \delta(x), x + \delta(x)) \subset [0, 1] \setminus F_{f,\varepsilon}^{\gamma+1},$$

we get  $y \notin F_{f,\varepsilon}^{\gamma+1}$ , which contradicts  $F_{f,\varepsilon}^\xi \subset F_{f,\varepsilon}^{\gamma+1}$ . So  $\gamma = \xi$ .

Again from the definition of  $\delta$ , there exists  $t \in F_{f,\varepsilon}^\gamma \setminus F_{f,\varepsilon}^{\gamma+1}$ , such that  $\delta^t(x) > \frac{3}{4}\delta(x)$ . So  $d(x, [0, 1] \setminus V_t) > \frac{3}{2}\delta(x) > |x - y|$ . Hence  $x, y \in V_t$ . From the definition of  $V_t$  it follows that  $|f(x) - f(y)| < \varepsilon$ .

(3) The function  $\delta$  restricted to the set  $F_{f,\varepsilon}^\gamma$  is continuous on the set  $F_{f,\varepsilon}^\gamma \setminus F_{f,\varepsilon}^{\gamma+1}$ .

Proof. The function  $\delta$  is defined on the set  $F_{f,\varepsilon}^\gamma \setminus F_{f,\varepsilon}^{\gamma+1}$  as an upper bound of the family of functions that fulfill the Lipschitz condition with common constant 1. Moreover the set  $F_{f,\varepsilon}^\gamma \setminus F_{f,\varepsilon}^{\gamma+1}$  is open with respect to  $F_{f,\varepsilon}^\gamma$ .

(4)  $\beta(\delta) \leq \alpha$ .

Proof. For a given  $\eta > 0$  let us consider the sequence of sets  $\{F_{\delta,\eta}^\gamma\}_{\gamma < \omega_1}$ . We shall show by the transfinite induction that for every  $\gamma < \omega_1$  we have

$$F_{\delta,\eta}^\gamma \subset F_{f,\varepsilon}^\gamma.$$

The inclusion is obvious for  $\gamma = 0$  because  $[0, 1] \subset [0, 1]$ . Suppose that for  $\gamma < \omega_1$  the above inclusion holds. From (3) it follows that the oscillation of the function  $\delta$  is equal to 0 in each point of the set  $F_{f,\varepsilon}^\gamma \setminus F_{f,\varepsilon}^{\gamma+1}$ . Hence  $F_{\delta,\eta}^{\gamma+1} \subset F_{f,\varepsilon}^{\gamma+1}$ . So  $\beta(\delta, \eta) \leq \beta(f, \varepsilon) \leq \alpha$ , and, as the number  $\eta$  is arbitrary, we have  $\beta(\delta) \leq \alpha$ .

(5) The function  $\delta$  is upper semicontinuous.

Consider  $x_0 \in [0, 1]$  and a sequence  $(x_n)_{n \in \mathbb{N}}$  convergent to  $x_0$  such that  $\lim_{n \rightarrow \infty} \delta(x_n) = g$ . We shall show that  $g \leq \delta(x_0)$ . There exists the ordinal  $\gamma < \alpha$ , such that  $x_0 \in F_{f,\varepsilon}^\gamma \setminus F_{f,\varepsilon}^{\gamma+1}$ . As  $x_0 \notin F_{f,\varepsilon}^{\gamma+1}$  and the set  $F_{f,\varepsilon}^{\gamma+1}$  is closed we can assume that no term of the sequence  $(x_n)$  belongs to  $F_{f,\varepsilon}^{\gamma+1}$ . There are two possibilities:

1° almost every term of the sequence  $(x_n)$  belongs to  $F_{f,\varepsilon}^\gamma \setminus F_{f,\varepsilon}^{\gamma+1}$ ;

2° infinitely many terms of that sequence belongs to  $[0, 1] \setminus F_{f,\varepsilon}^\gamma$ .

In the first case we have  $g = \lim_{n \rightarrow \infty} \delta(x_n) = \delta(x_0)$  as the function  $\delta$  is continuous on  $F_{f,\varepsilon}^\gamma \setminus F_{f,\varepsilon}^{\gamma+1}$ .

In the second case we can assume that all terms of  $(x_n)$  belong to  $[0, 1] \setminus F_{f,\varepsilon}^\gamma$ . For a given  $n$  in virtue of (1) we obtain

$$x_0 \notin (x_n - \delta(x_n), x_n + \delta(x_n)),$$

hence

$$|x_0 - x_n| \geq \delta(x_n) > 0,$$

and

$$g = \lim_{n \rightarrow \infty} \delta(x_n) = 0 < \delta(x_0).$$

As a result  $\delta$  is upper semicontinuous.

(6) The function  $\delta$  belongs to  $B1^*$ .

Let  $F \subset [0, 1]$  be an arbitrary nonempty closed set. We shall show existence of the interval  $(a, b)$  such that  $(a, b) \cap F \neq \emptyset$  and  $\delta|_F$  is continuous on  $(a, b) \cap F$ . Let  $\gamma$  be the least ordinal such that  $F \subset F_{f,\varepsilon}^\gamma$  but  $F \not\subset F_{f,\varepsilon}^{\gamma+1}$ . Let  $x_0 \in F \cap (F_{f,\varepsilon}^\gamma \setminus F_{f,\varepsilon}^{\gamma+1})$ . From (1) and (3), the function  $\delta$  is continuous on the set  $(x_0 - \delta(x_0), x_0 + \delta(x_0)) \cap F_{f,\varepsilon}^\gamma$ , and hence continuous on the set  $(x_0 - \delta(x_0), x_0 + \delta(x_0)) \cap F$ . As a consequence  $\delta \in B1^*$ .  $\square$

**Remark 1** The last result is similar (but not comparable) to the main result of Atok, Tang and Zhao ([4], Theorem 2). Nevertheless we decided to demonstrate our theorem because the proof is completely different.

## References

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