1. Let $X = \{1, 2, 3\}, \mathcal{C}_1 = \{\{1\}, \{2, 3\}, \emptyset, X\}$, and $\mathcal{C}_2 = \{\{3\}, \{1, 2\}, \emptyset, X\}$. Verify that $\mathcal{C}_1$ and $\mathcal{C}_2$ are both $\sigma$-algebras but $\mathcal{C}_1 \cup \mathcal{C}_2$ is not a $\sigma$-algebra.

2. Let $X$ and $Y$ be nonempty sets, and $(Y, \mathcal{F})$ be a measurable space. For any function $f : X \to Y$, show that the collection $\{f^{-1}(A) \mid \forall A \in \mathcal{F}\}$ is a $\sigma$-algebra in $X$.

3. Let $(X, \mathcal{C})$ be a measurable space. If the $\sigma$-algebra $\mathcal{C}$ has a finite number of subsets of $X$ in it, then $\mathcal{C}$ is also a topology on $X$.

4. Let $X = \{a, b, c, d\}$, $\mathcal{C} = \{X, \emptyset, \{a\}, \{b, c, d\}\}$ and let $\mathcal{F} = \mathcal{P}(X)$, the set of all subsets of $X$. Define the functions $f, g : X \to X$ by

$$f(x) = a \quad \text{for } x \in X$$

and

$$g(x) = \begin{cases} a & \text{if } x = a, b \\ c & \text{if } x = c, d. \end{cases}$$

Show that $f$ is $(\mathcal{C}, \mathcal{F})$-measurable but $g$ is not $(\mathcal{C}, \mathcal{F})$-measurable.

5. (a) Let $f : X \to \mathbb{R}$ be a constant function, $f(x) = \alpha$ for all $x \in X$. Show that $f$ is measurable.
   (b) Let $f : X \to \mathbb{R}$ be a measurable function and $\alpha \in \mathbb{R}$ be any real constant. Show that $\alpha f$ is also measurable.

6. Let $(X, \mathcal{C})$ be a measurable space and let $f, g : X \to \mathbb{R}$ be any two measurable real-valued functions. Show that the functions $f + g$, $f - g$ and $fg$ are measurable.

7. Define a relation on $[0, 1]$ as follows: for $x, y \in [0, 1]$ we say $x$ is related to $y$, written as $x \sim y$, if $x - y \in \mathbb{Q}$. Prove the followings:
(a) Show that \( \sim \) is an equivalence relation on \([0, 1]\).

(b) Let \( \{E_\alpha\}_{\alpha \in I} \) denote the set of equivalence classes of elements of \([0, 1]\). Using the axiom of choice, choose exactly one element \( x_\alpha \in E_\alpha \) for every \( \alpha \in I \) and construct the set \( E = \{x_\alpha \mid \alpha \in I\} \). Let \( r_1, r_2, r_3, \ldots, r_n, \ldots \) denote an enumeration of the rationals in \([-1, 1]\). Let \( E_n := r_n + E, \ n = 1, 2, 3, \ldots \)

Show that \( E_n \cap E_m = \emptyset \) for \( n \neq m \) and \( E_n \subseteq [-1, 2] \) for every \( n \). Deduce

\[
[0, 1] \subseteq \bigcup_{n=1}^{\infty} E_n \subseteq [-1, 2].
\]

(c) Show that \( E \) is not Lebesgue measurable.

8. Let \( X \) be a set and \( A, B, C \subset X \). The function \( \chi_A : X \to \{0, 1\} \) defined by

\[
\chi_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin A 
\end{cases}
\]
is called the characteristic (or indicator) function of \( A \). Show

(a) \( \chi_{A \cap B} = \chi_A \cdot \chi_B \), where \( \chi_A \cdot \chi_B(x) = \chi_A(x) \chi_B(x) \).

(b) \( \chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B} \).

9. Let \((X, C, \lambda)\) be a measure space, \( \psi \in M^+(X, C) \) be a simple function and \( c \geq 0 \) be a real number. Then show that

\[
\int c \psi \, d\lambda = c \int \psi \, d\lambda.
\]

10. Let \((X, C, \lambda)\) be a measure space, \( \psi, \xi \in M^+(X, C) \) be simple functions. Then show that

\[
\int (\psi + \xi) \, d\lambda = \int \xi \, d\lambda + \int \psi \, d\lambda.
\]

11. Let \((X, C, \lambda)\) be a measure space, and let \( \psi \in M^+(X, C) \) be a simple function, and \( A \in C \). Define \( \int_A \psi \, d\lambda := \int \psi \chi_A \, d\lambda \), where \( \chi_A \) is the characteristic function of the set \( A \). Show that the set function \( \mu(A) \) defined by

\[
\mu(A) = \int_A \psi \, d\lambda
\]
is a measure on the \( \sigma \)-algebra \( C \).
12. Let \((f_n)\) be a sequence of elements in \(M^+(X, \mathcal{C})\). By \(\lim \inf f_n\) we mean the function \(K\) defined by

\[ K(x) = \lim \inf f_n(x). \]

Prove the followings:

(1). [Fatou’s Lemma] Let \((f_n)\) be a sequence of elements in \(M^+(X, \mathcal{C})\) and let \(\lambda\) be a measure on \(\mathcal{C}\). Then

\[ \int (\lim \inf f_n) \, d\lambda \leq \lim \inf \int f_n \, d\lambda. \]

(2). Let \(f \in M^+(X, \mathcal{C})\) and let \(\lambda\) be a measure on \(\mathcal{C}\). If \(E = \{x \in X \mid f(x) > 0\}\) is an element of \(\mathcal{C}\) and \(\lambda(E) = 0\), then \(\int f \, d\lambda = 0\).

Hint: Define \(f_n(x) = n \chi_E(x)\). Show that this sequence satisfies the hypothesis of Fatou’s Lemma and that \(f \leq \lim \inf f_n\). Use Fatou’s Lemma and Theorem 2.2.20 to complete the proof.

13. Let \((X, \mathcal{C}, \lambda)\) be a measure space. Prove that the indicator function \(\chi_A\) is non-measurable, where \(A\) is a non-measurable set of \(X\).

14. Let \((X, \mathcal{C}, \lambda)\) be a measure space and let \(f_n : X \to \mathbb{R}, n = 1, 2, 3, \ldots\) be a monotonic sequence of measurable functions such that \(\lim f_n(x) = f(x)\). Prove that \(f(x)\) is measurable.

15. Let \(f(x)\) be 1, if \(x\) is a rational number, 0 otherwise. Prove that \(f(x)\) is not Riemann integrable on the interval \([0, 1]\).

September 28, 2010