1. Let $X$ be a random variable such that $E[|X|^n] < \infty$ for $n \in \mathbb{N}$. Let $\epsilon > 0$. Prove that
   \[ P(|X| \geq \epsilon) \leq \frac{E[|X|^n]}{\epsilon^n} \]

   Hint: Let $E = \{\omega : |X| \geq \epsilon\}$. Calculate $E[|X|^n]$ by using Definition (4.3.1), Theorem (4.3.10).

2. (Tchebychev's Inequality) Prove that if $E[X^2] < \infty$ then
   \[ P(|X-\mu| \geq \epsilon) \leq \frac{\text{Var } X}{\epsilon^2} \quad \text{for } \epsilon > 0. \]

3. Let $X$ be a random variable such that $\text{Var } (X)$ exists.
   Let $a$ be a constant. Show that
   \[ \text{Var } a(X) = a^2 \text{ Var } (X) \]

5. Find the density function $f$ for the discrete random variable $X$ defined in Exercise (4.3.4) and show that
   even though $\sum x f(x)$ in finite $E[X]$ does not exist.

6. Let $X$ be a random variable of the continuous type with density $f$. Show that if $E[X]$ exists then
   \[ \int_{-\infty}^{\infty} x f(x) \, dx \]
   is finite.
4. Show that Theorem (4.3.10) is false if the condition
\[ \sum_{i=1}^{\infty} \int_{E_i} |x| \, dP < \infty \]
is replaced by the condition
\[ \sum_{i=1}^{\infty} \int_{E_i} x \, dP, \text{ exists.} \]

Hint: Consider \((\Omega, \mathcal{F}, P)\) where \(\Omega = (0, 1)\) and \(P\) is the Lebesgue measure. Let
\[ E_i = \left( \frac{1}{i+1}, \frac{1}{i} \right), \quad i = 1, 2, 3, \ldots \]
Define \(X : \Omega \to \mathbb{R}\) by \(X(\omega) = (-1)^i (i+1)\) for \(\omega \in E_i\).
Show that even though
\[ \int_{E_i} x \, dP \]
exists for each \(i\) and
\[ \sum_{i=1}^{\infty} \int_{E_i} x \, dP \]
exists, \(\int x \, dP\) fails to exist by proving that
\[ \int |x| \, dP = E(|x|) \]
does not exist.

7. Prove that if \(E[X^n]\) exists for \(n\) a natural number \(E[X^r]\) exists for \(r\) a natural number, \(r \leq n\).

Hint: Note that \(|X^r| \leq 1 + |X^n|\) and apply Theorem (4.3.3).

8. Let \(X\) be a random variable with uniform distribution over \([a, b]\). Use moment generating function techniques to find \(E[X]\).

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