RESEARCH STATEMENT

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Overview

My research interests are inherently broad ranging and expanding, being driven by an openness to challenges of new problems and collaborators, as well as a fascination with the interaction between global and local combinatorial structure. Thus far my research has focused on two primary directions, the combinatorial properties of random graphs with an emphasis on those related to the spectrum, and the combinatorics of partially ordered sets. In this overview, I highlight how some of my recent work illuminates the interaction between local and global combinatorial structure. In the sections that follow, I provide a more detailed discussion of my research and how I see my research progressing without the intervention of new collaborative opportunities.

Perhaps the most recognized example of local structure influencing the global structure is the Hajnal-Szemeredí theorem [18], which gives a tight minimum degree condition to guarantee the existence of a K_r-factor in a graph. Recently, with Balogh, Lee, and Kemkes, I generalized this result to weighted graphs, providing upper and lower bounds on the minimum weighted degree necessary to assure that there is a K_r-factor with every clique having weight at least $t\binom{r}{2}$ [1]. As a surprising consequence of our work, it is likely that for fixed t the minimum degree condition is a decreasing function of r.

In the study of random graphs the expansion of the graph often plays a somewhat strange role as both a local and a global combinatorial property. For instance, the Alon-Boppana theorem gives bounds on the spectrum of the adjacency matrix based on the local condition that every vertex has a fixed degree d [33]. Recently, I extended this theorem to the spectrum of the normalized Laplacian for graphs with irregular degree sequences [50]. These bounds on the spectrum of the normalized Laplacian may be thought of as bounding how good of an expander can be constructed with a given degree sequence. As an additional consequence of my work, I show that for a fixed number of edges, a regular (or near regular) graph is not the best expander.

At other times the expansion properties of a graph may be thought of as a local property, in that any particular set of vertices has a relatively large neighborhood. Many of the results on flows on graphs take this viewpoint, showing that if a graph is a sufficiently good expander then there are "good" routing schemes (see, for instance [16, 17, 27, 49]). Ironically, my recent work with Chung and Zhao on selfish routing (in particular, Braess's paradox) in expanders [8] follows this scheme with the opposite conclusion; the presence of expansion allows for a selfish routing scheme to decrease the overall performance of the network.

The local combinatorial structure has a somewhat different flavor when considering the combinatorics of posets. For instance, in my recent work with Biró, Howard, Keller, and Trotter on partitioning a subposet of the subset lattice, our techniques were guided by specific cardinality relationships between successive levels of the subset lattice [2]. This work was inspired by a commutative algebra conjecture of R. Stanley from 1982 [44] and has since been cited by several papers within that community.

1. Spectral and Random Graph Theory

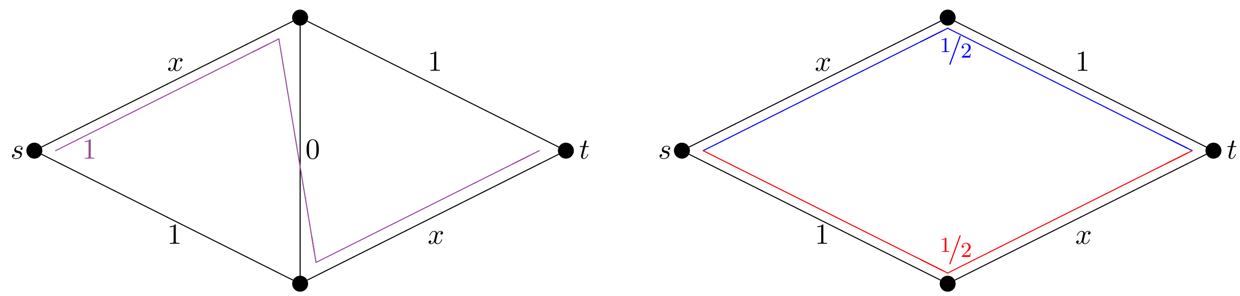
Although there are several different spectra associated with a random graph, my research focuses on the spectrum of the normalized Laplacian $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$ (where A is the adjacency matrix and D the diagonal matrix of degrees) which has been shown to be closely related to many algorithmic and structural aspects of the graph [9]. Many of these results, rather than relying on the full spectrum only use the spectral gap λ^*, which bounds the non-trivial eigenvalues in the interval $[\lambda^*, 2-\lambda^*]$. For example, if the spectral gap is large then there exist efficient, distributed, low-congestion methods for multi-commodity flow on the underlying network [16, 17, 27, 49]. For these results, the key structural observation is that the spectral gap is closely related to the expansion properties (conductance, Cheeger constant, etc.) of the underlying network [30, 43]. Because of the broad reaching applications of the spectrum of the normalized Laplacian, much of my work on random graphs has focused on bounding, calculating, and applying the spectrum or spectral gap.

The well-known Alon-Boppana theorem provides an explicit upper bound of ${}^{2\sqrt{d-1}}/_{d} + o(1)$ on the spectral gap of a d-regular graph, where the $o(1)$ term decreases with the diameter of the graph. Recently, by considering the weighted spectral radius of the universal cover graph, I generalized this result to graphs with non-uniform degrees, providing an upper bound of ${}^{2\bar{d}\sqrt{\bar{d}-1}}/_{\tilde{d}} + o(1)$, where $\bar{d}$ and $\tilde{d}$ are the average degree and the averaged squared degree, and the $o(1)$ term depends on the "robustness" of the graph [50]. Unless the graph is regular, this bound is strictly larger than the bound provided by Alon-Boppana, and in fact, I show that this difference is necessary. That is, for a given number of edges, the associated regular graph is not the extremal expander.

However, from an application point of view, an upper bound on the spectral gap is of limited use, and so there has been considerable effort expended in the explicit calculation/estimation of the spectral gap for various networks. In particular, with the continued rise in importance of complex networks such as the internet, there has been significant effort in analyzing the spectral gap of networks that may model complex networks (see [3, 6, 13, 14, 17, 31]). Unfortunately, these arguments are highly ad-hoc and there are few unifying themes between them. While developing an analysis for the semantic model for complex networks which appears in my thesis [51, 52, 54], Mihail and I derived a general reduction scheme for estimating the spectrum of the normalized Laplacian. Specifically, building off ideas used by Chung, Lu, and Vu to analyze the expected degree sequence model [6], we reduce the analysis of the spectrum to two matrices, one deterministic and one which is in expectation the zero matrix [32]. As a consequence, we improve the condition for the expected degree sequence model having at least constant spectral gap from $\omega(\ln^2(n))$ to $\Omega(\ln^{3/2}(n))$. It is also worth noting that, unlike recent matrix concentration techniques [34, 39], our techniques will apply to graphs with dependent edges.

As mentioned previously, one of the most powerful aspects of the expansion of networks is that it implies the existence of efficient, distributed, low-congestion methods for multi-commodity flow. However, implicit in these results is that the routing is occurring in a cooperative manner, which anyone who has ever sat through rush hour traffic knows is not realistic in all situations. Thus, it is natural to consider what can be said in a selfish routing situation. Perhaps one of the most unusual results regarding selfish routing is Braess's observation that there are theoretical road networks where it is possible to improve the common latency by removing roads, even roads with extremely fast travel times. Specifically, he considered selfishly routing one unit flow from s to t in the network depicted in Figure 1. When the flow is routed selfishly (see Figure 1(a)), all of the flow passes through the zero latency central edge and the common latency is 2. However, by removing the central zero latency edge (see Figure 1(b)), the selfish routing will spread the flow uniformly over the paths between s and t, resulting in an common latency of $3/2$.

Since its discovery, Braess's paradox has spawned a significant amount of work aimed at understanding the full implications of the paradox, both theoretically [15, 22, 36, 37] and via anecdotal observations [12, 28]. In many ways the trend towards studying the "Price of Anarchy" [29, 35] has its roots in the discovery of Braess's paradox. However, these "worst case" analyses via designer instances give little insight into the practical consequences of Braess's paradox. Although the anecdotal evidence indicates that Braess's paradox can occur in real world networks, it gives little to no feeling for how prevalent or severe the paradox it is in real world networks. Recently, Valiant and Roughgarden [48] began to answer this question by providing the first proof that Braess's paradox



(a) Original Network with equilibrium latency 2

(b) Modified network with equilibrium latency $^3/_2$

Figure 1: Braess's Paradox

could occur in a large class of non-designer graphs. Specifically, they showed that in sufficiently dense instances of Erdős-Rényi random graphs with affine latency functions satisfying certain mild conditions, Braess's paradox occurs with high probability. Chung and I were able to show that this phenomenon extends almost down to the connectivity threshold for Erdős-Rényi random graphs [7]. With Zhao, we have extended our analysis to encapsulate expanders (with no global minimum degree condition) and latency functions which need only be convex [8]. Ironically, this implies that precisely those conditions (such as, expansion) that are known to lead to good routing behaviors also inexorably lead to Braess's paradox.

1.1. **Future Directions.** The recent work of Oliveira [34] and Chung and Radcliffe [39] has essentially resolved the question of determining the spectrum for random graphs with independent edges. However, many recent models of random graphs, especially those developed to model complex networks, are introducing a two-step random process where the edge probabilities themselves are random (see, for instance, the multiplicative attribute graphs [26] and my work on random dot product graphs [54, 53, 51]). Together with Radcliffe, I am working on resolving the issues posed by these random graphs and developing a general framework for their analysis.

Even if the case of the two-step random process could be resolved, there would still be a large class of graphs to be dealt with, namely those with dependent edges. By my decomposition result, this can be dealt with by bounding the spectral norm of a matrix C where $\mathbb{E}[C] = 0$. Since $\mathbb{E}[C] = 0$, the spectral norm of C may be thought of as a centered random variable in a non-commutative probability space, which brings the tools of free probability into play. As an added benefit, much of the combinatorial structure for non-commutative probability spaces comes from the lattice of non-crossing partitions, and thus lies in the intersection of my current research interests.

With Chung and some of her graduate students, I have recently begun considering the application of expanders/spectral graph theory to pursuit-evasion games, otherwise known as Cops and Robbers. The primary open question in this field is how many cops are necessary to catch a single robber on a graph. It has been conjectured by Meyniel that on an n vertex graph the number of cops necessary is $\mathcal{O}(\sqrt{n})$, however the current results (found independently by several groups) only show that $\mathcal{O}\big(n^{1-o(1)}\big)$ suffices. As all of these results rely on enforcing a very slight expansion condition, we are exploring how strengthening this condition affects the number of cops required.

2. Combinatorics of Partially Ordered Sets

My research involving the combinatorics of partially ordered sets has focused primarily on two questions: First, what is the nature of the interval partitions of the "closed" subposets of the sub(multi)set lattice? And second, how "unfair" is the act of forming a linear extension of a poset?

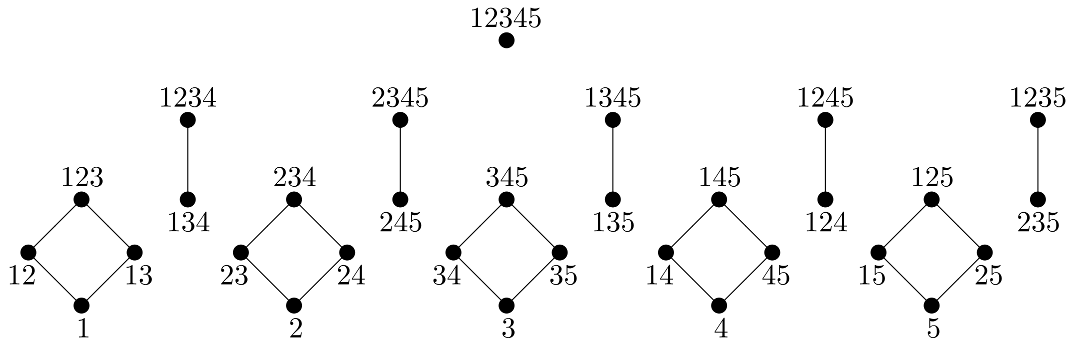


Figure 2: Partition of $\mathbf{2}^5 - \varnothing$ into intervals witnessing Stanley depth 3

2.1. **Poset Partitioning.** The first of these questions arises naturally from the work of Herzog et al. on Stanley depth, an invariant from commutative algebra which is at the heart of a conjecture that is almost 30 years old [44]. More formally, the combinatorial question is; given a multi-set $\mathcal{G}$ on n symbols and an upwardly closed collection collection of sub-multi-sets, what is the largest k such that the collection can be partitioned into intervals $[A, B] = \{C \colon A \subseteq C \subseteq B\}$ where B is maximal in at least k symbols? The work of Herzog et al. shows that this quantity is the Stanley depth of a particular module encoded by the instance. In the case where $\mathcal{G}$ is a set (corresponding to the square-free monomial ideals), the question can be rephrased as: Given an anti-chain $\mathcal{A}$ in the subset lattice on n elements, how large can k be so that the upset of $\mathcal{A}$ can be partitioned into intervals $[A_i, B_i]$ so that $|B_i| \geq k$ for all i. As part of the their work Herzog et al. found by computer experiment that if the antichain is $\{\varnothing\}$, then for $n \leq 9$ the maximal value of k is $\lceil \frac{n}{2} \rceil$.

With Biró, Howard, Keller, and Trotter, I was able to use purely combinatorial techniques to finish the work of Herzog et al. on $\mathbf{2}^n - \{\varnothing\}$, showing that $k = \lceil n/2 \rceil$ [2]. We provided two separate proofs of this result by constructing partitions (one recursively and one directly) of $\mathbf{2}^{2t+1} - \varnothing$ with the surprising additional property that every interval $[A, B]$ satisfies that $|A| = 2s + 1$ and $|B| = s + t + 1$. For example, see the partition into intervals for $t = 2$ given in Figure 2. This result provides, to our knowledge, the first calculation of the Stanley depth for an infinite class of modules, namely the maximal ideals.

From a strictly combinatorial point of view, there are two natural directions to generalize this result: The first is to consider the partitions to consider the subposet of the subset lattice where every element has size at least d. The second is to consider a lower bound on k based on the number of minimal elements. In fact, for the first, Shen was able to extend our methods to show that if $n \leq d < 4d + 3$, then the value for k is $\lfloor n-d/d+1 \rfloor + d$ [42]. For the second, Shen and Cimpoeaş (independently) used our work as a base case to show that a for a special class of minimal elements, if there are m minimal elements then $k \geq n - \lfloor m/2 \rfloor$ [10, 42]. Using local combinatorial techniques my coauthors and I were able to significantly shorten both of these proofs and extend their applicability. Specifically, for the first, Keller, Shen, Streib, and I were able to extend the range of applicability to $d \leq n < 5d + 4$ [23] while simplifying the proofs. While for the second, Keller and I showed, via a significantly shorter proof, that restriction on the minimal elements was unnecessary [24].

2.2. **Poset Unfariness.** The definition of "unfairness" of a linear extension in my second area of research was introduced by Tanenbaum et al. and termed linear discrepancy [45]. One of the motivations for their study was the following hypothetical scenario: patients are triaged on their arrival into an emergency room and are treated according to the seriousness of their injuries. Naturally, one would wish to treat patients with incomparable injuries as close together as possible while still respecting the ordering imposed by the injuries. For a given poset, the least k so that there is a linear extension in which incomparable elements are separated by at most k is the linear discrepancy of the poset. It is worth noting that the linear discrepancy of a poset is equal to the bandwidth of its co-comparability graph [11]. At the end of their founding work, Tanenbaum et al. proposed a series of eight challenges and questions for future work on linear discrepancy, which have been the focus of my research in this area.

The first of these questions was to provide a characterization of the linear discrepancy two posets. Initial progress was made by Rautenbach who provided a list of 20 posets he conjectured provided a forbidden poset characterization of linear discrepancy at most two [41]. Howard et al. showed that in fact there are an infinite number of such forbidden posets of width two [19]. Howard, Keller, and I finished this list giving a complete forbidden subposet characterization of the poset of linear discrepancy at most two [20].

The second question Tanenbaum et al. asked essentially concerns when the ability to treat in parallel incomparable injuries reduces the "unfairness." Using the work of Choi and West [5], Howard and I provided complete characterization of those posets where parallel processing does not mitigate the unfairness. However, despite this characterization we were also able to show that recognizing such posets is $\mathcal{NP}$-complete [21].

Fundamental to the methods for both of these results is a careful analysis of how a given linear discrepancy affects the allowable incomparability sets of elements. My remaining work on this question focused on how restrictions on the number of incomparabilities of an element affects the linear discrepancy. For instance, Tanenbaum et al. speculated that if every element is incomparable to at most t elements, then the linear discrepancy is at most $\lfloor 3t-1/2 \rfloor$. Together with Keller, I provided partial confirmation for this conjecture, showing that it holds for disconnected posets [25]. Additionally, we were able to prove a Brooks's-type theorem for the linear discrepancy of interval orders, showing that if every element in an interval order is incomparable to at most t elements then the linear discrepancy is at most t. Further, the inequality is strict unless the interval order contains an antichain of size $t + 1$.

2.3. **Future Directions.** Currently, with Keller, I am looking at how the poset partitioning question is effected by the complementation map. Specifically, given an instance of the poset partitioning problem, what can be said about the problem on the complement (within the natural lattice defined by the multi-set)? This question is motivated by recent observations [38, 40] that the separation of the answers to these partitioning questions implies Stanley's conjecture for the class of monomial ideals. We have preliminary results indicating that when $\mathcal{G}$ has few elements this separation occurs.

Although the motivation for the poset partitioning problem has its roots in relatively longstanding conjectures in commutative algebra, it leads to a host of natural combinatorial questions, such as: What is the complexity of calculating the Stanley depth? What is the Stanley depth of a "random" monomial ideal or its quotient? Does the combinatorial structure uniquely encode the algebraic structure? Is there a general relationship between the minimal elements of the poset and the Stanley depth of the poset?

I believe the most intriguing open question regarding linear discrepancy is the question posed by Brightwell and Trotter [11, 47]: If $\mathbf{P}$ has dimension at least 5, is it the case that $\dim(\mathbf{P}) \leq \mathrm{ld}(\mathbf{P})$? This conjecture is motivated by the observation that $\dim(\mathbf{P}) \leq \mathrm{width}(\mathbf{P})$ [46], whereas $\mathrm{ld}(\mathbf{P}) \geq \mathrm{width}(\mathbf{P})$. My work with Keller [25] indicates one possible approach to this problem, through the context of the critical pairs. Specifically, we show that, like dimension, linear discrepancy is fundamentally about the behavior of critical pairs, giving a possible route to attack this conjecture.

Recently I have also been working with Jeff Remmel to explore the consequences of the bijection between unlabelled interval orders and ascent sequences [4]. Specifically, we are looking at the nature of this bijection when restricted to semiorders. Our preliminary results have pointed to a surprising potential structure theorem for the canonical representation of semiorders.

References

[1] Jozsef Balógh, Graeme Kemkes, Choongbum Lee, and Stephen J. Young, *Towards a weighted version of the Hajnal-Szemerédi theorem.* submitted, August 2011.

[2] Csaba Biró, David M. Howard, Mitchel T. Keller, William T. Trotter, and Stephen J. Young, *Interval partitions and Stanley depth*, J. Combin. Theory Ser. A, 117 (2010), pp. 475–482.

[3] Stefan Bornholdt and Heinz Georg Schuster, eds., *Handbook of graphs and networks*, Wiley-VCH, Weinheim, 2003. From the genome to the internet.

[4] Mireille Bousquet-Mélou, Anders Claesson, Mark Dukes, and Sergey Kitaev, $(2+2)$*-free posets, ascent sequences and pattern avoiding permutations*, J. Combin. Theory Ser. A, 117 (2010), pp. 884–909.

[5] Jeong-Ok Choi and Douglas B. West, *Forbidden subposets for fractional weak discrepancy at most k*, European Journal of Combinatorics, (2010), p. *to appear*.

[6] Fan Chung, Linyuan Lu, and Van Vu, *The spectra of random graphs with given expected degrees*, Internet Math., 1 (2004), pp. 257–275.

[7] Fan Chung and Stephen J. Young, *Braess's paradox in large sparse random graphs*, in 6th Workshop on Internet and Network Economics, 2010.

[8] Fan Chung, Stephen J. Young, and Wenbo Zhao, *Braess's paradox in expanders.* submitted, September 2011.

[9] Fan R. K. Chung, *Spectral graph theory*, vol. 92 of CBMS Regional Conference Series in Mathematics, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1997.

[10] Mircea Cimpoeaş, *Stanley depth of complete intersection monomial ideals*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), 51(99) (2008), pp. 205–211.

[11] Peter C. Fishburn, Paul J. Tanenbaum, and Ann N. Trenk, *Linear discrepancy and bandwidth*, Order, 18 (2001), pp. 237–245.

[12] Caroline Fisk and Stefano Pallottino, *Empirical evidence for equilibrium paradoxes with implications for optimal planning strategies*, Transportation Research Part A: General, 15 (1981), pp. 245 – 248.

[13] Abraham D. Flaxman, Alan M. Frieze, and Juan Vera, *A geometric preferential attachment model of networks*, Internet Math., 3 (2006), pp. 187–205.

[14] ———, *A geometric preferential attachment model of networks ii*, in Algorithms and Models for the Web-Graph, Fan Chung and Anthony Bonato, eds., vol. 4863 of Lecture Notes in Computer Science, 2007, pp. 41–55.

[15] Marguerite Frank, *The Braess paradox*, Math. Programming, 20 (1981), pp. 283–302.

[16] Alan M. Frieze, *Edge-disjoint paths in expander graphs*, SIAM J. Comput., 30 (2001), pp. 1790–1801 (electronic).

[17] Christos Gkantsidis, Milena Mihail, and Amin Saberi, *Conductance and congestion in power law graphs*, SIGMETRICS Perform. Eval. Rev., 31 (2003), pp. 148–159.

[18] A. Hajnal and E. Szemerédi, *Proof of a conjecture of P. Erdős*, in Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969), North-Holland, Amsterdam, 1970, pp. 601–623.

[19] David M. Howard, Gab-Byung Chae, Minseok Cheong, and Sang-Mok Kim, *Irreducible width 2 posets of linear discrepancy 3*, Order, 25 (2008), pp. 105–119.

[20] David M. Howard, Mitchel T. Keller, and Stephen J. Young, *A characterization of partially ordered sets with linear discrepancy equal to 2*, Order, 24 (2007), pp. 139–153.

[21] David M. Howard and Stephen J. Young, *When linear and weak discrepancy are equal.* in preperation, 2009.

[22] H. Kameda, *How harmful the paradox can be in the braess/cohen-kelly-jeffries networks*, in INFOCOM 2002. Twenty-First Annual Joint Conference of the IEEE Computer and Communications Societies. Proceedings. IEEE, vol. 1, 2002, pp. 437 – 445 vol.1.

[23] Mitchel T. Keller, Yi-Huang Shen, Noah Streib, and Stephen J. Young, *On the Stanley depth of squarefree Veronese ideals*, Journal of Algebraic Combintorics, (2010). accepted.

[24] Mitchel T. Keller and Stephen J. Young, *Stanley depth of squarefree monomial ideals*, Journal of Algebra, 322 (2009), pp. 3789 – 3792.

[25] Mitchel T. Keller and Stephen J. Young, *Degree bounds for linear discrepancy of interval orders and disconnected posets*, Discrete Mathematics, 310 (2010), pp. 2198 – 2203.

[26] Myunghwan Kim and Jure Leskovec, *Multiplicative attribute graph model of real-world networks.* preprint, arXiv:1009.3499v3.

[27] Jon Kleinberg and Ronitt Rubinfeld, *Short paths in expander graphs*, in 37th Annual Symposium on Foundations of Computer Science (Burlington, VT, 1996), IEEE Comput. Soc. Press, Los Alamitos, CA, 1996, pp. 86–95.

[28] Gina Kolata, *What if they closed 42d street and nobody noticed?*, The New York Times, December 25 (1990).

[29] Elias Koutsoupias and Christos Papadimitriou, *Worst-case equilibria*, in 16th Annual Symposium on Theoretical Aspects of Computer Science, vol. 1563 of Lecture Notes in Comput. Sci., Springer Berlin, 1999, pp. 404–413.

[30] Gregory F. Lawler and Alan D. Sokal, *Bounds on the L^2 spectrum for Markov chains and Markov processes: a generalization of Cheeger's inequality*, Trans. Amer. Math. Soc., 309 (1988), pp. 557–580.

[31] Milena Mihail, Christos Papadimitriou, and Amin Saberi, *On certain connectivity properties of the internet topology*, J. Comput. System Sci., 72 (2006), pp. 239–251.

[32] Milena Mihail and Stephen J. Young, *On the spectral gap of inhomogeneous random graphs*. in preperation.

[33] A. Nilli, *On the second eigenvalue of a graph*, Discrete Math., 91 (1991), pp. 207–210.

[34] Roberto Imbuzeiro Oliveira, *Concentration of the adjacency matrix and of the laplacian in random graphs with independent edges*. preprint, arXiv:0911.0600v2.

[35] Christos Papadimitriou, *Algorithms, games, and the internet*, in Proceedings of the Thirty-Third Annual ACM Symposium on Theory of Computing, New York, 2001, ACM, pp. 749–753 (electronic).

[36] Eric I. Pas and Shari L. Principio, *Braess' paradox: Some new insights*, Transportation Research Part B: Methodological, 31 (1997), pp. 265 – 276.

[37] Claude M. Penchina, *Braess paradox: Maximum penalty in a minimal critical network*, Transportation Research Part A: Policy and Practice, 31 (1997), pp. 379 – 388.

[38] Dorin Popescu and Muhammad Imran Qureshi, *Computing the Stanley depth*, J. Algebra, 323 (2010), pp. 2943–2959.

[39] Mary Radcliffe and Fan Chung, *On the spectra of general random graphs*. preprint.

[40] Asia Rauf, *Depth and Stanley depth of multigraded modules*, Comm. Algebra, 38 (2010), pp. 773–784.

[41] Dieter Rautenbach, *A note on linear discrepancy and bandwidth*, J. Combin. Math. Combin. Comput., 55 (2005), pp. 199–208.

[42] Yi-Huang Shen, *Monomial Curves, Gorenstein Ideals and Stanley Decompositions*, PhD thesis, Purdue University, 2009.

[43] Alistair Sinclair and Mark Jerrum, *Approximate counting, uniform generation and rapidly mixing Markov chains*, Inform. and Comput., 82 (1989), pp. 93–133.

[44] Richard P. Stanley, *Linear Diophantine equations and local cohomology*, Invent. Math., 68 (1982), pp. 175–193.

[45] Paul J. Tanenbaum, Ann N. Trenk, and Peter C. Fishburn, *Linear discrepancy and weak discrepancy of partially ordered sets*, Order, 18 (2001), pp. 201–225.

[46] William T. Trotter, *Combinatorics and partially ordered sets*, Johns Hopkins Series in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, MD, 1992. Dimension theory.

[47] ———, *Research problems [Discrete mathematics], problem 383*, Discrete Math., 257 (2002), pp. 599–624. Kleitman and combinatorics: a celebration (Cambridge, MA, 1999).

[48] Gregory Valiant and Tim Roughgarden, *Braess's paradox in large random graphs*, Random Structures & Algorithms, 37 (2010), pp. 495–515.

[49] Vijay V. Vazirani, *Approximation algorithms*, Springer-Verlag, Berlin, 2001.

[50] Stephen J. Young, *The weighted spectrum of the universal cover and an Alon-Boppana result for the normalized Laplacian*. in preperation.

[51] ———, *Random Dot Product Graphs: A Flexible Model for Complex Networks*, PhD thesis, Georgia Institute of Technology, 2008.

[52] Stephen J. Young and Milena Mihail, *Random dot product graphs as models for complex networks*. in preperation.

[53] Stephen J. Young and Edward Scheinerman, *Directed random dot product graphs*, Internet Math., 5 (2008), pp. 91–111.

[54] Stephen J. Young and Edward R. Scheinerman, *Random dot product graph models for social networks*, in Algorithms and models for the web-graph, vol. 4863 of Lecture Notes in Comput. Sci., Springer, Berlin, 2007, pp. 138–149.