A Brooks-Type Theorem for the Bandwidth of Interval Graphs

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Abstract

Let G be an interval graph. The layout that arranges the intervals in order by right endpoint easily shows that the bandwidth of G is at most its maximum degree Δ. Hence, if G contains a clique of size $\Delta + 1$, then its bandwidth must be Δ. In this paper we prove a Brooks-type bound on the bandwidth of interval graphs. Namely, the bandwidth of an interval graph is at most Δ, with equality if and only if it contains a clique of size $\Delta + 1$. Furthermore, the stronger bound is tight even for interval graphs of clique number 2. Our proof utilizes the correspondence between the linear discrepancy of a partially ordered set and the bandwidth of its co-comparability graph. We also make progress toward a related question of Tanenbaum, Trenk, and Fishburn. They asked if a poset in which each point is incomparable to at most Δ others has linear discrepancy at most $\lfloor (3\Delta - 1)/2 \rfloor$. We show that this is true if the poset is disconnected.

Key words:
bandwidth, interval graph, linear discrepancy, interval order, poset
2000 MSC: 05C78, 05C62, 06A07

1. Introduction

The *bandwidth* of a graph G on n vertices is the least integer k such that there exists a labelling of the vertices by $[n] = \{1, 2, \ldots, n\}$ so that the labels of adjacent vertices differ by at most k. In general, calculating the bandwidth of a graph is NP-hard, even for trees with maximum degree 3 as shown in [1]. In fact,

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Preprint submitted to a journal *March 30, 2009*

Blache, Karpinski, and Wirtgen show that even for trees there is no polynomial time approximation scheme for calculating the bandwidth [2]. The difficulties in even calculating the bandwidth in the general case have led to a host of work on bounding and calculating the bandwidth for specific classes of graphs. (See [3, 4] for a survey of such results.) In this work, we will focus on providing degree-based bounds for the bandwidth of co-comparability graphs. To this end, we consider the following alternative formulation of the bandwidth of a graph.

Definition 1 (Bandwidth)**.** Let G be a graph on n vertices and $L\colon V(G) \to [n]$ be a bijection. Then define the bandwidth of the layout L, denoted $\mathrm{bw}(G, L)$, as $\max_{\{u,v\}\in E(G)} |L(u) - L(v)|$. The bandwidth of G, denoted $\mathrm{bw}(G)$, is the minimum of $\mathrm{bw}(G, L)$ over all layouts L.

If instead of graphs, the fundamental objects under consideration are posets, we would have a similar quantity—linear discrepancy. More specifically, if we denote by $h_L(x)$ the height of the element x in the linear extension L, we have the following analogous definition:

Definition 2 (Linear Discrepancy)**.** For a poset P and a linear extension L we denote the linear discrepancy of L as $\mathrm{ld}(P, L)$, and define

$$\mathrm{ld}(P, L) = \max_{x \|_P y} |h_L(x) - h_L(y)| \,.$$

The linear discrepancy of P, denoted $\mathrm{ld}(P)$, is the minimum of $\mathrm{ld}(P, L)$ over all linear extensions L.

Examining these two definitions, it is clear that if G is the co-comparability graph of a poset P, then $\mathrm{bw}(G) \le \mathrm{ld}(P)$, as every linear extension is a layout. In fact, if P is a poset and G is the associated co-comparability graph, Fishburn, Tanenbaum, and Trenk showed that $\mathrm{ld}(P) = \mathrm{bw}(G)$ [5]. Furthermore, they showed that given a poset P and its co-comparabiliy graph G, there is a polynomial time transformation that turns a layout of G that is optimal with respect to bandwidth into a linear extension of P that is optimal with respect to linear discrepancy and vice versa. This relationship was first exploited by Rautenbach to show, via observations regarding linear extensions, that for a co-comparability graph G, $\mathrm{bw}(G) \le 2\Delta(G) - 2$. In this paper we further exploit the relationship between bandwidth and linear discrepancy to prove two degree-based bounds on bandwidth. In particular, we derive a series of properties of linear extensions that are optimal with respect to linear discrepancy.

The main result of this paper, obtained by proving the analogous result for linear discrepancy, is the following Brooks-type theorem:

Theorem 1. *The bandwidth of an interval graph is at most its maximum degree Δ, with equality if and only if the graph contains a clique of size $\Delta + 1$.*

We also give an example demonstrating that Theorem 1 is tight even for interval orders of width 2.

For general co-comparability graphs, the equivalence of bandwidth and linear discrepancy trivially yields that the bandwidth is at most $2\Delta - 1$. The best known bound, due to Rautenbach [6], is that the bandwidth is at most $2\Delta - 2$. However, it is not known whether this bound is tight. In fact, inspired by their observation that $\mathrm{ld}(\mathbf{a_1} + \mathbf{a_2} + \cdots + \mathbf{a_t}) = \sum_{i=1}^{t} a_i - 1 - \max_i \lfloor a_i/2 \rfloor$, Tanenbaum, Trenk, and Fishburn conjectured that the linear discrepancy of a poset (respectively, the bandwidth of a co-comparability graph) is at most $\lfloor (3\Delta - 1)/2 \rfloor$. In Theorem 8 we are able to that the conjecture is true for the class of disconnected posets. However, in contrast to the naturally dual classes of interval orders and interval graphs, this class of posets has no nice analogue among co-comparability graphs. It is also worth noting that this result trivially implies that there is some class of connected posets for which the conjecture holds. For example, the conjecture holds for the class of posets consisting of the linear sum of disconnected posets. However, we do not see any means of generalizing the proof of Theorem 8 to a more interesting class of connected posets.

We will use the following notation. If x and y are incomparable elements of a poset P, we will write $x \parallel_P y$ or $x \parallel y$ if the poset P is clear. The set of all points incomparable to x will be denoted $\mathrm{Inc}(x)$. In order to better illustrate the relationship between a poset P and its co-comparability graph, we will define $\Delta(P) = \max_{x \in P} |\mathrm{Inc}(x)|$, which is the maximum degree in the co-comparability graph of P. When it is clear which poset is under consideration we will simply use Δ for $\Delta(P)$. If P and Q are posets with disjoint point sets, we will denote by $P + Q$ the disjoint union of P and Q. Furthermore, if n is a positive integer, $\mathbf{n}$ will represent the chain on n elements. A *linear extension* L of a poset P is a linear order on the points set of P such that if $x <_P y$, then $x <_L y$. We denote the down-set $\{y \in P \mid y < x\}$ by $D(x)$. The up-set $U(x)$ is defined dually. We will freely abuse this notation by treating subsets of the ground set of a poset as the subposet they induce. We say (x, y) is a *critical pair* in P if $x \|_P y$, $D(x) \subseteq D(y)$, and $U(y) \subseteq U(x)$. For any unfamiliar poset terminology or notation we refer the reader to [7].

2. Degree Bounds for Interval Orders

Given a co-comparability graph G, there can be several different posets with G as their co-comparability graph. However, if G is an interval graph then its interval representation induces a natural interval order that has co-comparability graph G. Although [8] shows that problem of determining the bandwidth of a co-comparability graph is NP-complete, in [9], Kleitman and Vohra were able to provide an efficient algorithm for determining whether the bandwidth of an interval graph is at most k. Combining this algorithm with binary search yields an algorithm for calculating the bandwidth of interval graphs, and hence the linear discrepancy of interval orders, in polynomial time. The essential idea behind the algorithm is to place the intervals from left to right while maintaining linear discrepancy at most k. We will use a similar idea to prove that, in general, the linear discrepancy of an interval order is at most Δ.

Without loss of generality, we will assume that all interval orders are presented via a unique endpoint representation. For an interval x, we will denote the right endpoint by $r(x)$ and the left endpoint by $\ell(x)$. The following theorem shows that, much like Brooks' Theorem, there is a natural upper bound on the linear discrepancy of an interval order in terms of Δ and that this bound can be improved by one if the poset does not contain an antichain of size $\Delta + 1$ (respectively, a clique of size $\Delta + 1$ for interval graphs) . However, we will also show that in general this bound cannot be decreased beyond $\Delta - 1$.

Theorem 2. *An interval order has linear discrepancy at most Δ, with equality if and only if it contains an antichain of size* $\Delta + 1$.

Proof. We note that it is implicit in the work of Fomin and Golovach [10] (via a pathwidth argument), that the bandwidth of an interval graph is at most Δ. However, we observe that the ordering of the vertices according to right endpoints yields a linear extension L with linear discrepancy at most Δ. This is because if $x \| y$, any element placed between x and y in L must also be incomparable to both x and y, meaning that there are at most $\Delta - 1$ elements placed between them and thus $|h_L(x) - h_L(y)| \leq \Delta$. If $\mathrm{width}\,(P) = \Delta + 1$, then trivially $\mathrm{ld}(P) \geq \Delta + 1 - 1 = \Delta$, so we must have $\mathrm{ld}(P) = \Delta$. The remainder of the proof shows that if this is not the case, we can strengthen the upper bound.

Let P be an interval order that does not contain an antichain of size $\Delta + 1$. By induction, we may assume that P cannot be partitioned into sets D and U such that $d < u$ for all $d \in D$ and $u \in U$, as otherwise $\mathrm{ld}(P) = \max\{\mathrm{ld}(D)\,, \mathrm{ld}(U)\}$. Fix an interval representation of P in which all endpoints are distinct and let m be the interval with smallest right endpoint and m' the interval with largest left endpoint. We may assume that m' also has the largest right endpoint. (Since m' must be maximal, we may do this by extending m' to the right.)

Now form a linear extension L of P by ordering the intervals by right endpoint. Let x be an aribtrary interval in $P - \{m, m'\}$. Now since P cannot be partitioned as $D \cup U$ with $d < u$ for all $d \in D$ and all $u \in U$, x must overlap an interval having larger right endpoint. Dually, x must overlap an interval with smaller right endpoint. Therefore, we must have an element of $\mathrm{Inc}(x)$ that precedes x in L and another that follows x in L. For every $x \in P$, the linear extension L has the property that the elements of $\mathrm{Inc}(x) \cup \{x\}$ appear consecutively, and thus for any $x \notin \{m, m'\}$ and y incomparable to x, we have $|h_L(x) - h_L(y)| \leq \Delta - 1$, since there are elements of $\mathrm{Inc}(x)$ on both sides of x.

It only remains to address the intervals m and m' with smallest right and largest left endpoints, respectively. Note that m is incomparable only to minimal elements and m' is incomparable only to maximal elements by our choice of representation. Since the minimal elements of P are an antichain and $\mathrm{width}\,(P) \leq \Delta$, m is incomparable to at most $\Delta - 1$ points, and thus $|h_L(x) - h_L(m)| \leq \Delta - 1$ for all x incomparable to m. A similar argument applies to m'. Therefore, $\Delta - 1 \geq \mathrm{ld}(P, L) \geq \mathrm{ld}(P)$. □

By the equivalence of the linear discrepancy of a poset with the bandwidth of co-comparability its graph, we may state this result in terms of the bandwidth of interval graphs, yielding the more natural-sounding statement of Theorem 1.

By considering the poset formed by adding one cover to an antichain on $\Delta + 1$ points (i.e., $\mathbf{2} + \mathbf{1} + \mathbf{1} + \cdots + \mathbf{1}$) it is clear that the bound provided in Theorem 2 is tight. However, in this case the tightness is a consequence of the trivial lower bound on the linear discrepancy of one less than the width. In order to show that this upper bound is nontrivial, we produce for each Δ an infinite family of width-two interval orders that have linear discrepancy $\Delta - 1$. The following two lemmas restricting the class of linear extensions that need to be considered will be helpful in establishing the linear discrepancy of the constructed posets.

Lemma 3. *For any linear extension L of a poset P, the maximum distance in L between incomparable elements is achieved at a critical pair.*

Proof. Suppose x and y are such that $x <_L y$ and achieve the maximum distance between incomparable elements in L. Suppose $x' <_P x$, then $x' <_L x$. But then by the maximality of (x, y), $x' <_P y$ and hence $D(x) \subseteq D(y)$. Similarly, $U(y) \subseteq U(x)$. Thus (x, y) is a critical pair. □

Lemma 4. *There exists a linear discrepancy optimal extension where no critical pair (x, y) is reversed unless (y, x) is also a critical pair.*

Proof. Let (x, y) be a critical pair such that (y, x) is not a critical pair. Thus either there is some $z \in D(y)$ such that z is incomparable to x, or there is some $z \in U(x)$ such that z is incomparable to y. Without loss of generality we

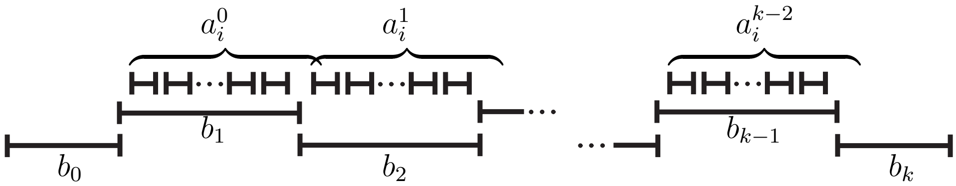

Figure 1: The interval order $\mathbf{F}_k^t$

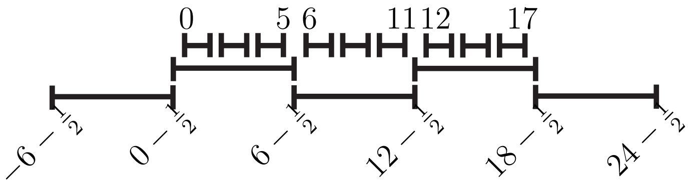

Figure 2: The interval order $\mathbf{F}_4^3$

may assume that $z \in D(y)$ and z is incomparable to x. Now suppose L is an extension of P that reverses (x, y). Then, in part L has the following relations, $D(y) <_L y <_L \cdots <_L x <_L U(x)$. Consider now the linear extension L' formed by exchanging x and y in L. By construction, everything between x and y in L is incomparable to both x and y, so L' is a linear extension. Furthermore, everything less than y in L and incomparable to L is also incomparable to x, and thus the interchange of x and y does not affect these distances in L'. In addition, the distance between z and y has decreased by at least one. $\square$

Thus equipped, we will define a family of interval orders $\{\mathbf{F}_k^t\}_{k\geq 3}^{t\geq 1}$ and show that for $k > t$, we have $\mathrm{ld}(\mathbf{F}_k^t) = \Delta(\mathbf{F}_k^t) - 1$. For each $t \geq 1$ and $k \geq 3$ define the elements of the interval order $\mathbf{F}_k^t$ as follows:

- For $0 \leq i \leq t-1$ and $0 \leq j \leq k-2$, the interval $[2jt + 2i, 2jt + 2i + 1]$ is the element a_i^j.
- For $0 \leq j \leq k$, the interval $\left[2(j-1)t - \frac{1}{2}, 2jt - \frac{1}{2}\right]$ is the element b_j.

Fig. 1 illustrates the interval representation of a general $\mathbf{F}_k^t$, while Fig. 2 shows $\mathbf{F}_4^3$. Note that $|\operatorname{Inc}(a_i^j)| = 1$ for all i, j, $|\operatorname{Inc}(b_j)| = t + 2$ for $1 \leq j \leq k-1$, and $|\operatorname{Inc}(b_0)| = |\operatorname{Inc}(b_k)| = 1$, so $\Delta(\mathbf{F}_k^t) = t + 2$. We also observe that $\operatorname{width}(\mathbf{F}_k^t) = 2$.

Proposition 5. *The linear discrepancy of* $\mathbf{F}_k^t$ *is at least* $t + 1 - \lfloor t/k \rfloor = \Delta - 1 - \lfloor (\Delta - 2)/k \rfloor$.

Proof. First we observe that the only critical pairs in $\mathbf{F}_k^t$ are of the form (b_i, b_{i+1}) and further, the remaining points form a chain of height $t(k-1)$. By Lemma 4, we may assume that in any linear extension L that is optimal extension with respect to linear discrepancy, L orders the b_i by index. Further, by Lemma 3 the distances between these pairs of points completely determine the linear discrepancy. Thus, we wish to distribute the $t(k-1)$ remaining points as equally as possible between the $k+1$ gaps among the elements $\{b_0, b_1, \ldots, b_k\}$. This results in one gap containing at least $\lceil t(k-1)/k \rceil = t - \lfloor t/k \rfloor$ elements, implying $\mathrm{ld}(\mathbf{F}_k^t) \geq t + 1 - \lfloor t/k \rfloor$. □

We note that in particular this implies that for any $k > t$, we have $\Delta - 1 = t + 1 \leq \mathrm{ld}(\mathbf{F}_k^t) \leq \Delta - 1$ and so Theorem 2 is tight, even for posets of width 2.

3. Linear Discrepancy of Disconnected Posets

In [11], the concept of k-discrepancy irreducibility as a linear discrepancy analogue of irreducibility of dimension is introduced. This concept has been used to some effect in [12, 13] to provide (in conjunction with the work of Tanenbaum, Trenk and Fishburn in [14]) a complete forbidden subposet characterization of posets with linear discrepancy at most two. However, it has not been shown whether linear discrepancy irreducibility is truely analogous to dimension irreducibility in that it was not known whether having linear discrepancy at least k assured the existence of a k-discrepancy irreducible subposet, except in the cases of $k = 1, 2$ as shown in [14] and $k = 3$ as shown in [13]. In contrast to dimension, by considering removing the isolated point from $\mathbf{1} + \mathbf{n}$, it is clear that the removal of a single point can decrease the linear discrepancy by an arbitrarily large amount. However, the following lemma answers the general question in the affirmative by designating a specific point whose removal reduces the linear discrepancy by at most one.

Lemma 6. *For any poset there exists a point whose removal reduces the linear discrepancy by at most one.*

Proof. First suppose that there are two minimal elements x and x' of P with the same up-set. Let L be a linear linear extension of $P - \{x'\}$ optimal with respect to linear discrepancy. Create an new extension L' by inserting x' immediately below x in L. It is clear that L' is a linear extension of P. Furthermore since $\mathrm{Inc}(x) - \{x'\} = \mathrm{Inc}(x') - \{x\}$, the linear discrepancy of L' is at most one more than the linear discrepancy of L. Thus the removal of x decreased the discrepancy of P by at most one.

Now suppose that there are no two minimal elements with the same up-set. Then there is a minimal emement z that is never the top element of a critical pair.

Specifically there does not exist $z' \in P$ with $U(z') \subset U(z)$. Consider a linear extension S of $P - \{z\}$ that is optimal with respect to linear discrepancy and let s be the minimal element of $U(z) \cup \{v \mid (z, v) \in \text{crit}\,(P)\}$ with respect to S. Create a linear extension S' by inserting z immediately below s. By construction, S' is an linear extension of P. Since we only wish to show that the linear discrepancy of S' is at most one more than the linear discrepancy of S, the only obstructions are of the form $z \parallel z'$. But by Lemma 3 and the choice of z, we may restrict our attention to critical pairs (z, z').

If $s \in U(z)$, then $s \parallel z'$ since otherwise z and z' are comparable. If $z \notin U(z)$, then (z, s) is a critical pair, so $U(s) \subseteq U(z)$ and in particular $s \parallel z'$. But then, $h_{S'}(z') - h_{S'}(z) = h_S(z') - h_S(s) + 1 \leq \text{ld}(P - \{z\}) + 1$. Hence the linear discrepancy of $P - \{z\}$ is at least $\text{ld}(P) - 1$. □

Corollary 7. *If* $\text{ld}(P) \geq k$*, then* P *contains an induced* k*-discrepancy irreducible subposet.*

Tanenbaum, Trenk and Fishburn conjectured in [14] that for any poset the linear discrepancy is at most $\lfloor (3\Delta - 1)/2 \rfloor$. Their conjecture was based on their observations of the linear discrepancy of the sum of chains, in particular that $\text{ld}(\mathbf{d} + \mathbf{d}) = \lfloor (3d - 1)/2 \rfloor$. We use Lemma 6 to show that this bound holds for disconnected posets while leaving the more general case open.

Theorem 8. *A disconnected poset has linear discrepancy at most* $\left\lfloor \frac{3\Delta - 1}{2} \right\rfloor$.

Proof. We proceed by contradiction. Suppose P is the minimal counterexample, and hence discrepancy-irreducible. Fix $\Delta = \Delta(P)$. Now suppose there is some isolated point $x \in P$. Then $\text{ld}(P) \leq |P| - 1 \leq |\text{Inc}(x)| = \Delta$. Thus removing a single point from P maintains that P is disconnected. In particular, since removing a single point does not increase Δ, we have by minimality $\text{ld}(P) = \lfloor (3\Delta - 1)/2 \rfloor + 1$. Further, let Q be a subposet with $\text{ld}(Q) = \text{ld}(P) - 1$ formed by removing a minimal element of P as guaranteed by Lemma 6 and the irreducibility of P. Suppose that $\Delta(Q) \leq \Delta - 1$. Then

$$\left\lfloor \frac{3\Delta - 1}{2} \right\rfloor = \text{ld}(Q) \leq \left\lfloor \frac{3\Delta(Q) - 1}{2} \right\rfloor \leq \left\lfloor \frac{3\Delta - 4}{2} \right\rfloor = \left\lfloor \frac{3\Delta - 2}{2} \right\rfloor - 1 < \left\lfloor \frac{3\Delta - 1}{2} \right\rfloor .$$

Thus $\Delta(Q) = \Delta$.

Now Q is disconnected, so let (A, B) be a partition of the elements witnessing this with $|A| \leq |B|$. We observe that $\Delta(A) \leq \Delta - |B|$ and $\Delta(B) \leq \Delta - |A|$.

Let L_B be an optimal linear extension of B. Form the linear extension L of Q by taking the first $\lceil |B|/2 \rceil$ elements of L_B, followed by all of A, then the last $\lfloor |B|/2 \rfloor$ elements of L_B. Then $\mathrm{ld}(Q) \leq \mathrm{ld}(Q, L)$, and so in particular,

$$\left\lfloor \frac{3\Delta - 1}{2} \right\rfloor \leq \max\left\{ |A| + \left\lceil \frac{|B|}{2} \right\rceil - 1, |A| + \mathrm{ld}(B) \right\}.$$

Suppose first that $\lfloor (3\Delta - 1)/2 \rfloor \leq |A| + \mathrm{ld}(B)$. Now $\mathrm{ld}(B) \leq 2\Delta(B) - 2$ by [6]. Combining this with the observation that $\Delta(B) \leq \Delta - |A|$, we have $|A| \leq 2\Delta - \lfloor (3\Delta - 1)/2 \rfloor - 2$. Since $\mathrm{ld}(Q) = \lfloor (3\Delta - 1)/2 \rfloor$, we have that $|A| + |B| \geq \lfloor (3\Delta - 1)/2 \rfloor + 1$. Thus, $|B| \geq 2\lfloor (3\Delta - 1)/2 \rfloor + 3 - 2\Delta \geq \Delta + 1$, a contradiction.

Now suppose that $\lfloor (3\Delta - 1)/2 \rfloor \leq |A| + \lceil |B|/2 \rceil - 1$. Thus

$$\left\lfloor \frac{3\Delta - 1}{2} \right\rfloor \leq |A| + \left\lceil \frac{|B|}{2} \right\rceil - 1 = \left\lceil \frac{3|B| - 2}{2} \right\rceil \leq \left\lceil \frac{3\Delta - 3\Delta(A) - 2}{2} \right\rceil.$$

Hence $\Delta(A) = 0$ and $|B| \leq \Delta$. Similarly,

$$\left\lfloor \frac{3\Delta - 1}{2} \right\rfloor \leq \left\lceil \frac{2|A| + |B| - 2}{2} \right\rceil \leq \left\lceil \frac{3\Delta - 2\Delta(B) - 2}{2} \right\rceil = \left\lceil \frac{3\Delta - 2}{2} \right\rceil - \Delta(B).$$

Hence $\Delta(B) = 0$, and Q is the sum of two chains. But then by a lemma in [14], the linear discrepancy of Q is $|A| + \lceil |B|/2 \rceil - 1$, and so $|A| = |B| = \Delta$. It is then easy to see that there is no valid way to create P from Q such that $\mathrm{ld}(P) > \lfloor (3\Delta - 1)/2 \rfloor$. Thus if P is a disconnected poset, $\mathrm{ld}(P) \leq \lfloor (3\Delta - 1)/2 \rfloor$. □

4. Conclusions and Future Work

Although the tightness of Theorem 2 and the algorithm of Kleitman-Vohra would seem to completely resolve the questions of linear discrepancy of interval orders/bandwidth of interval graphs, the techniques used bring up a host of questions. Perhaps the most intriguing direction for future work would be to explore the relationship between linear discrepancy and dimension through their dependence on critical pairs. We see no intuitive reason for the relationship between linear discrepancy and critical pairs and so it is possible that the relationship is simply a happy coincidence. However, if there were an intuitive explanation for this relationship, it would perhaps suggest a proof of the conjecture that if $\mathrm{ld}(P) = \dim P = n \geq 5$, then P contains the standard example $\mathbf{S}_n$ as a subposet. (See [14, 15].) Since the class of interval orders contains posets of arbitrarily large dimension but not the standard examples for $n > 1$, it would be interesting to see if this conjecture can be proven for this restricted class of posets.

We have also left open the question of whether $\mathrm{ld}(P) \leq \lfloor (3\Delta(P) - 1)/2 \rfloor$ for connected posets. There are few results stated only for connected posets, and for good reason. Adding a new element greater than all the elements in a poset yields a connected poset, and for most combinatorial questions, this does not change anything. The proof of Theorem 8 as well as the motivation for the conjecture itself hinge fundamentally on the large number of comparabilities in disconnected posets. We see no reason to believe that the conjecture is not true in general, but at the same time we do not see how our methods could be extended. Any improvement to the best known bound of $2\Delta(P) - 2$, such as a result of the form $\mathrm{ld}(P) \leq (2 - \varepsilon)\Delta(P)$, would be very welcome. In fact, even the question of whether the conjecture is true for $\Delta = 4$ (i.e., whether the correct upper bound is 5 or 6) is open, and perhaps an answer to this question would give additional insight into the larger problem.

Acknowledgments

The authors are particularly grateful to David M. Howard for many useful conversations involving linear discrepancy, especially his involvement in the discovery of Lemmas 3 and 4. William T. Trotter also patiently listened to the ideas of this paper, especially the highly unusual result of Theorem 8.

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