POSETS WITH COVER GRAPH OF PATHWIDTH TWO HAVE BOUNDED DIMENSION

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Abstract. Joret, Micek, Milans, Trotter, Walczak, and Wang recently asked if there exists a constant d such that if P is a poset with cover graph of P of pathwidth at most 2, then $\dim(P) \leq d$. We answer this question in the affirmative. We also show that if P is a poset containing the standard example S_5 as a subposet, then the cover graph of P has treewidth at least 3.

1. Introduction

Although the dimension of a poset and the treewidth of a graph have been prominent subjects of mathematical study for many years, it is only recently that the impact of the treewidth of graphs on poset dimension has received any real attention. This new interest in connections between these topics has led to recasting an old result in terms of treewidth. It is natural to phrase the following 36-year-old result in terms of treewidth, which had been defined (using a different name) by Halin in [4] a year earlier. However, the importance of treewidth (and the use of that name) only became widely known through the work of Robertson and Seymour [8] nearly a decade later.

Theorem 1.1 (Trotter and Moore [13]). *If $P = (X, \leq)$ is a poset such that the cover graph of P is a tree, then $\dim(P) \leq 3$. Equivalently, if P is a poset having connected cover graph of treewidth at most 1, then $\dim(P) \leq 3$.*

Recently there have been a number of papers on the dimension of planar posets [2, 3, 10]. This work naturally led to the question of bounding a poset's dimension in terms of the treewidth of its cover graph. Over 30 years ago, Kelly showed in [6] that there are planar posets having arbitrarily large dimension by constructing a planar poset containing S_d, the standard example of dimension d, as a subposet. These examples use large height to stretch out S_d to allow a planar embedding. Joret et al. [5] point out that the pathwidth of Kelly's examples is 3 for $d \geq 5$. Thus, any bound on dimension solely in terms of pathwidth or treewidth is impossible. However, they were able to show that it suffices to add a bound on the height in order to bound the dimension. In particular, they prove the following:

Theorem 1.2 (Joret et al. [5]). *For every pair of positive integers (t, h), there exists a least positive integer $d = d(t, h)$ so that if P is a poset of height at most h and the treewidth of the cover graph of P is at most t, then $\dim(P) \leq d$.*

Motivated by the observation about the pathwidth of Kelly's examples, Joret et al. conclude their paper by asking if there is a constant d such that if P is a poset

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whose cover graph has pathwidth at most 2, then $\dim(P) \leq d$. They also ask this question with treewidth replacing pathwidth. (An affirmative answer to the latter question would imply an affirmative answer to the former.) In this paper, we show that the answer for pathwidth 2 is in fact "yes" with the following result:

Theorem 1.3. *Let P be a poset. If the cover graph of P has pathwidth at most 2, then $\dim(P) \leq 26$.*

In fact, the precise version of this result (Theorem 4.3) is intermediate between answering the pathwidth question and answering the treewidth question, as we only need to exclude six of the 110 forbidden minors that characterize the graphs of pathwidth at most 2. (Treewidth at most 2 is characterized simply by forbidding K_4 as a minor.)

We show in Theorem 5.2 that any poset containing the standard example S_5 has treewidth at least 3. Trotter [11] has raised the question of whether a planar poset with large dimension must contain a large standard example. If the answer to Trotter's question is "yes", combining this with our Theorem 5.2 would yield our Theorem 1.3 as well as a bound on the dimension of posets with cover graphs having treewidth at most 2.

Before proceeding to our proofs, we provide some definitions for completeness. We then establish some essential properties of the 2-connected blocks of a graph of pathwidth at most 2. We then prove the more general version of Theorem 1.3 and conclude with the rather technical proof that posets containing S_5 have cover graphs of treewidth at least 3.

2. Definitions and Pathwidth 2 Obstructions

Let $P = (X, \leq)$ be a poset. If $x < y$ in P and there is no $z \in X$ such that $x < z < y$, we say that x is covered by y (or y covers x) and write $x <: y$. The cover graph of P is a graph G with vertex set X and x is adjacent to y in G if and only if $x <: y$ or $y <: x$. (If we view the order diagram of P as a graph, that graph is P's cover graph.) The dimension of P is the least t such that there exist t linear extensions $L_1, \ldots, L_t$ of P with the property that $x < y$ in P if and only if $x < y$ in L_i for $i = 1, \ldots t$. If x and y are incomparable in P, every $z \leq x$ in P is also less than y in P, and every $w \geq y$ in P is also greater than x in P, then we say that (x, y) is a critical pair of P.

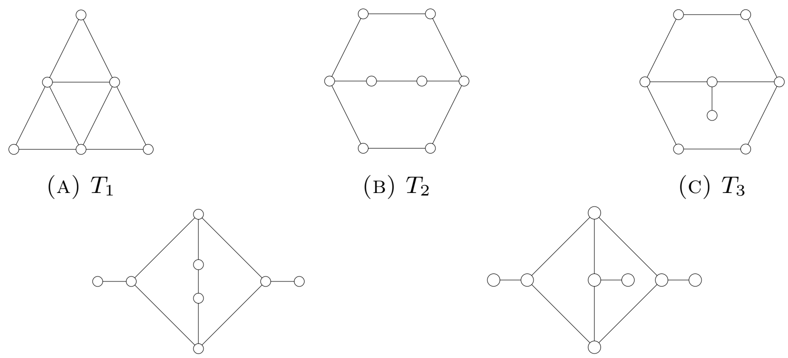
Felsner, Trotter, and Wiechert [3] showed the following result on the dimension of posets with outerplanar cover graphs, which will prove essential to our proof.

Theorem 2.1 (Felsner, Trotter, Wiechert [3])**.** *If a poset P has outerplanar cover graph, then $\dim(P) \leq 4$.*

Let $G = (V, E)$ be a graph. A pair $(T, \mathcal{V})$, where T is a tree and $\mathcal{V} = (V_t)_{t \in T}$ with $V_t \subseteq V$ for all $t \in T$, is a *tree-decomposition* of G if

1. $V(G) = \cup_{t \in T} V_t$;
2. for every $e \in E$, there exists a vertex t of T such that $e \subseteq V_t$; and
3. if t_1, t_2, t_3 are vertices of T and t_2 lies on the unique path from t_1 to t_3 in T, then $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$.

The sets V_t are often referred to as the *bags* of the tree-decomposition. The width of $(T, \mathcal{V})$ is $\max_t |V_t| - 1$. The treewidth of G, which we denote by $\mathrm{tw}(G)$ is the minimum width of a tree-decomposition of G. A *path-decomposition* of a graph is



(A) T_1 (B) T_2 (C) T_3 (D) T_4 (E) T_5

FIGURE 1. Five key obstructions for pathwidth 2.

a tree-decomposition in which the tree T is a path. The pathwidth of G, denoted $\text{pw}(G)$, is the minimum width of a path-decomposition of G.

If G is a graph and H is a subgraph of G, we say that a path P is an H-path if P is nontrivial and intersects H precisely at its two end vertices. The length of a path is the number of edges it contains.

The set of graphs of pathwidth at most k is a minor closed family. Therefore, by the Graph Minor Theorem [9], there exists a finite set of obstructions. For $k = 2$, Kinnersley and Langston found the entire set of 110 obstructions in [7]. Although it is useful to have the whole list at hand, our proof will require only six graphs from it. Besides the obvious obstruction K_4, the others are depicted in Figure 1. It is elementary to verify that these graphs have pathwidth 3. We will refer to these graphs in the proof by the names shown.

3. Properties of the 2-connected blocks

We begin without restricting our attention to only cover graphs. In this section, we consider a graph G such that $\text{pw}(G) \leq 2$ and prove strong properties about the block structure. This structure is essential in the proof of our main theorem. To establish this structural result, we first make the following definition.

Definition 3.1. A *parallel nearly outerplanar graph* is a graph that consists of a longest cycle C with vertices labelled (in order) as $x_1, x_2, \ldots, x_k, y_l, y_{l-1}, \ldots, y_1$ along with some chords and chords subdivided exactly once. The chords and subdivided chords have attachment points $x_{i_1}, y_{j_1}, \ldots, x_{i_m}, y_{j_m}$ such that $i_1 \leq \cdots \leq i_m$ and $j_1 \leq \cdots \leq j_m$.

An example of a parallel nearly outerplanar graph is shown in Figure 2. We think of the vertices along the bottom of the cycle as being the x_i and those along the top as being the y_j. Vertices to the left of the leftmost chord and to the right of the rightmost chord could be either x_i's or y_j's.

Lemma 3.2. *A graph G is a parallel nearly outerplanar graph if and only if G is 2-connected and* $\text{pw}(G) \leq 2$.

Proof. It is easy to see that every parallel nearly outerplanar graph is 2-connected and has pathwidth at most 2. A path decomposition of pathwidth 2 can be obtained

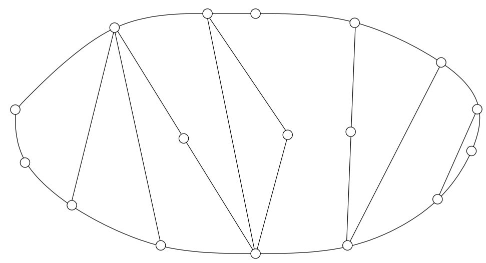


FIGURE 2. A parallel nearly outerplanar graph.

by starting with the bag containing x_1 and y_1 and working through the x_i and y_j by increasing subscript. Replace x_i or y_j by x_{i+1} or y_{j+1}, respectively, after all chords incident with it have had their other attachment point included in a bag with it. The internal vertex of a subdivided chord appears in a bag with precisely its two attachment points.

For the converse, let C be a longest cycle in G. A C-path will be called an *ear*. We first note that C can not have crossing ears. More precisely, if P and Q are ears, $V(C) \cap V(P) = \{p_1, p_2\}$, and $V(C) \cap V(Q) = \{q_1, q_2\}$, then the order of these intersection vertices on C must be $p_i, p_{2-i}, q_j, q_{2-j}$ for some $i, j = 1, 2$. If this were not the case, then G would have a K_4 minor, forcing $\mathrm{pw}(G) \geq 3$.

Next we show that no ear may have more than one internal vertex. Indeed, if P is an ear with at least two internal vertices and $V(C) \cap P = \{v_1, v_2\}$, then both paths between v_1 and v_2 on C must contain at least two internal vertices, for otherwise C is not the longest cycle. If this occurs, then G has T_2 as a minor.

We now show that the internal vertex of any ear is of degree 2. Let v be the internal vertex of the ear xvy. Let H be the subgraph induced by the vertices of C and the vertex v. If v has degree at least 3 in H, then H contains a K_4 minor. Otherwise there is a $v' \in V(G)$ such that $v' \sim v$, but $v' \notin V(H)$. Let H' be the subgraph of G induced by $V(H) \cup \{v'\}$. Since G is 2-connected and H' is not, there is an H'-path P (possibly just a single edge) with one endpoint being v'. The other endpoint may only be x or y, otherwise we have a K_4 minor. Without loss of generality, the other endpoint is x, which implies that $xPv'vy$ is an ear with at least two internal vertices, a contradiction.

We have now shown that G contains a (longest) cycle and some non-crossing ears with at most one inner vertex which must have degree two. The only thing that remains to be shown is that the vertices of the cycle may be labeled as in the definition, effectively placing an ordered structure on the ears. If this were not true, there would be three ears with attachment points a_1, b_1, a_2, b_2, and a_3, b_3 that appear around the longest cycle of G ordered as $a_1, b_1, a_2, b_2, a_3, b_3$ around C, with the possibility that $b_i = a_{i+1}$ for any i (cyclically). In this case, G contains the forbidden subgraph T_1 as minor, giving our final contradiction. □

We note that after proving Lemma 3.2, we discovered that Barát et al. [1] proved this lemma independently while working to simplify the characterization of graphs of pathwidth 2. They used the name *track* for what we call a parallel nearly outerplanar graph. We use the latter name because it is more evocative of the aspects of the structure that are important in our proof.

By Lemma 3.2, each 2-connected block of a graph of pathwidth two is a parallel nearly outerplanar graph. Our next lemma establishes that the vertices where these blocks join together lie on the parallel nearly outerplanar graphs' longest cycles.

Lemma 3.3. *Let G be a connected graph with $\mathrm{pw}(G) \leq 2$. Let B be a 2-connected block of G. If there is a vertex v of B adjacent to a vertex v' not in B, then $v \in V(C)$ where C is a longest cycle of B.*

Proof. Suppose that $v \notin V(C)$. This means that v must be an internal vertex of an ear xvy. Deletion of x and y from the cycle C leaves two arcs (paths), which we will call C_1 and C_2. If both C_1 and C_2 contain at least two vertices, then G has T_3 as a minor, since we have assumed that v has a neighbor v' not in B. Thus, suppose C_1 contains a single vertex u. If the degree of u in G is two, then the cycle C' formed from by replacing u by v is also a longest cycle of B, and so the result holds with C' in place of C. Therefore, we may assume that u is adjacent to a vertex u' not in B. Furthermore, $u' \neq v'$, and there is no path from u' to v' in G that does not go through B. If C_2 contains at least two vertices, then G contains T_4 as a minor. If C_2 is a single vertex w, then it must have degree 2 in G to avoid having T_5 as a minor. But then the lemma holds by considering $\{x, v, y, u\}$ as the longest cycle and xwy as an ear. □

4. Posets with Cover Graphs of Pathwidth 2

Definition 4.1. Let P be a poset. A *subdivison* of the cover relation $x <: y$ in P is the addition of new points $x_1, x_2, \dots, x_k$ such that $x < x_1 < \cdots < x_k < y$ and the new points x_i are incomparable with all points of P that are not greater than y or less than x. We say that Q is a *subdivision* of P if Q can be constructed from P by subdividing some of its cover relations.

Lemma 4.2. *Let P be a poset and Q be a subdivision of P. Then* $\dim(Q) \leq 2\dim(P)$.

Proof. Let $\{L_1, \dots, L_d\}$ be a realizer of P. For each L_i, we construct two linear extensions L_i' and L_i'' of Q. For each subdividing chain $x < x_1 < \cdots < x_k < y$ of an original cover $x <: y$, we place the x_i consecutively in order immediately above x in L_i'. We form L_i'' by placing the x_i consecutively in order immediately below y. It is easy to verify that $\{L_1', L_1'', \dots, L_d', L_d''\}$ is a realizer of Q. □

Now we are ready to prove the main theorem. Let G be a parallel nearly outerplanar graph that is the cover graph of a poset P, and call its longest cycle C. An ear with no inner vertex is simply called a *chord*. We call an ear xyz *unidirected* if $x < y < z$ or $z < y < x$ in P. Otherwise we call it a *beak*. An *upbeak* is an ear with $x < y > z$ in P, and a *downbeak* is an ear with $x > y < z$ in P. We call the internal point of a beak a *beak peak*.

Theorem 4.3. *Let P be a poset with cover graph G. If G does not contain K_4 and the graphs T_i for $i = 1, \dots, 5$ as shown in Figure 1 as minors, then* $\dim(P) \leq 26$.

Proof. Consider the block decomposition of G. For each block B, remove the points of P corresponding to the internal vertices of unidirected ears of B. The poset we obtain after doing this for each block will be called Q. By Lemma 4.2, it suffices to show that $\dim(Q) \leq 13$.

We now form a poset R from Q by removing any point that coresonds to a beak peak of a two-connected block of the cover graph of Q. It is clear that the cover graph of R is outerplanar, so Theorem 2.1 implies $\dim(R) \leq 4$. To construct a realizer of Q, we will use a realizer of R as the basis for constructing a set of four linear extensions and then devise two further sets of four linear extensions to reverse the remaining critical pairs. Those twelve linear extensions will combine with one extra linear extension to realize Q.

Let $\{L'_1, L'_2, L'_3, L'_4\}$ be a realizer of R. From L'_i, we construct a linear extension L_i of Q by inserting the beak peaks. We insert the downbeak peaks as high as possible in L'_i and the upbeak peaks as low as possible in L'_i. It is easy to verify that these linear extensions of Q will reverse every critical pair that does not involve a beak peak. In addition, if xyz is a downbeak, and w is such that (y, w) is a critical pair and w is comparable to x or z, then (y, w) is reversed. To see why, suppose $w < x$. Then in one of the L'_i, we must have $w < x < z$. In this case, inserting y as high as possible reverses (y, w). Symmetrically, if xyz is an upbeak, (w, y) is a critical pair, and w is comparable to x or z, then (w, y) is reversed.

We now use a single linear extension to reverse all critical pairs of the form (w, y), where xyz is a downbeak, as well as those of the form (y, w), where xyz is an upbeak. To do this, let L_0 be a linear extension formed from L'_1 by adding the downbeak peaks at the bottom and the upbeak peaks at the top. We order the beak peaks opposite to how they are ordered in L_1 to ensure all critical pairs involving two beak peaks of the same type are reversed as well. It is clear that this accomplishes our stated goal.

The only critical pairs remaining to reverse are those involving the peak of a beak xyz and a vertex incomparable to both x and z. To construct linear extensions reversing these critical pairs, we first define two (typically not linear) extensions Q_1 and Q_2 of Q. Our general approach is to successively consider beaks xyz. At each step, we add a relation between x and z and then take transitive closure before proceeding to the next beak whose attachment points remain incomparable. We will have two rules for determining whether $x < z$ or $x > z$, resulting in two different extensions.

We begin by describing the process used to construct the extension Q_1; the process to construct Q_2 will be a small variation. To construct Q_1, we consider the 2-connected blocks of the cover graph of Q one at a time. Since two blocks intersect only at a single point on their longest cycles, introducing a new comparability within one block cannot force two incomparable beak attachment points in another block to become comparable by transitivity. Therefore, the order in which we consider the blocks does not matter.

Consider a 2-connected block B. Since B is a parallel nearly outerplanar graph, a fixed plane embedding provides (up to duality) a natural left-to-right ordering on its beaks as suggested in Figure 2. Fix one of these orders and number the k beaks of B accordingly from 1 to k. Denote the attachment points for beak i by x_i and z_i. We will define the extension Q_1 by considering the points x_i and z_i in order for $i = 1, \ldots, k$. At each stage, if x_i and z_i have not already been made comparable by transitivity, we specify a method for determining the order which they should be placed.

We begin by (arbitrarily) making $x_1 > z_1$ in Q_1. We now suppose that $i > 1$ and we wish to establish a relation between x_i and z_i, with the relationship between x_j

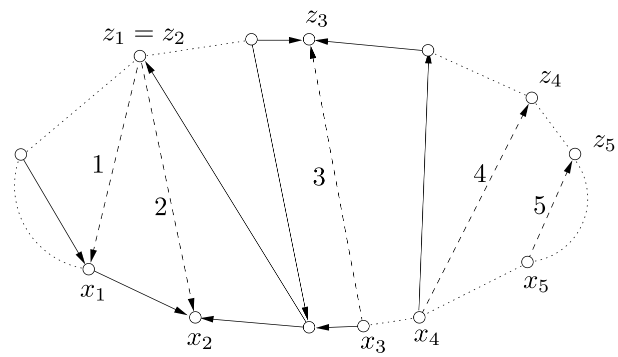


FIGURE 3. Orientations added for Q_1. Solid arrows are comparabilities (going up), dotted lines indicate paths with internal vertices omitted that are explicitly *not* directed, and dashed lines are beaks. Beak 1 receives an arbitrary orientation, which forces the orientation of beak 2 by transitivity. Then $S(x_3) = 1$ and $S(z_3) = 3$, which determines the orientation of beak 3. Similarly, $S(x_4) = 3$ and $S(z_4) = 4$, so $x_4 < z_4$. Since $S(z_5) = S(x_5) = 5$, and $z_5 \| x_5$, beak 5 was oriented arbitrarily.

and z_j for $j < i$ already having been determined. For $u = x_i$ or $u = z_i$, define

$$S(u) = \min\{j : u \leq x_j \text{ or } u \leq z_j\}.$$

If $S(x_i) \neq S(z_i)$, then the relation between x_i and z_i in Q_1 agrees with the relation between $S(x_i)$ and $S(z_i)$. If $S(x_i) = S(z_i)$, define the relation between x_i and z_i in Q_1 arbitrarily. Before proceeding to another pair of beak attachment points, we take the transitive closure. We then repeat the process for the pair of beak attachment points with lowest index that have not already been made comparable.

The process used to define Q_2 is similar to that for Q_1. The difference is that we begin by arbitrarily making x_k and z_k comparable and then consider the beaks in decreasing order of index. We also change "minimum" in the definition to S to "maximum."

Notice that Q_1 and Q_2 have outerplanar cover graphs. This is because if xyz were a beak in Q, the addition of a comparability between x and z would mean that the vertex corresponding to y in the cover graph would be only adjacent to one of x and z. In this case, y could be moved to the outside face and the new edge connecting x and z could follow the path formerly determined by the edges xy and yz. For the extensions Q_1 and Q_2 to be of use to us, we must have that any critical pairs not reversed by one of the linear extensions $L_0, \ldots, L_4$ remain incomparable in one of Q_1 and Q_2. More precisely, we have the following:

Claim. Let xyz be a beak in Q and suppose that w is incomparable to x, y, z in Q. Then w is incomparable to y in one of Q_1 or Q_2.

Proof of claim. Let B be a 2-connected block of the cover graph of Q and let xyz be an upbeak in B. Suppose that w is incomparable to x, y, z in Q. We will show that $w \not< x$ and $w \not< z$ in at least one of Q_1 or Q_2. This suffices as it would be the only way to have made y and w comparable when xyz is an upbeak. The proof for the case where xyz is a downbeak will be omitted, as it is symmetric.

First, suppose that $w \in B$ and w is not a beak peak. Recalling the natural left-to-right order used to construct Q_1 and Q_2, we assume that w is right of the beak xyz and consider Q_1. Suppose for a contradiction that $w < x$ or $w < z$ in Q_1. For brevity, we write $w < u$ where $u \in \{x, z\}$. Since w is incomparable to u in Q, there must have been a step in the process of extending Q that made $w < u$ for the first time. Suppose this was created by taking the transitive closure after determining the relation between the beak attachment points x' and z', where we determined $x' < z'$. Consider the moment before we added this relation. For transitivity to subsequently force $w < u$, at this point we must have $w \leq x'$ and $z' \leq u$. We first observe that x' and z' must be right of x and z, as otherwise the beak structure would require that w be less than x or z in order to have $w \leq x'$. Now the process for constructing the extension Q_1 dictates that before adding $x' < z'$, we must have $S(x') \leq S(z')$. For this to happen with $z' \leq u \in \{x, z\}$, the left-to-right ordering now forces x' to be less than or equal to either x or z. But then w was comparable to x or z *before* we added the relation, a contradiction. The argument when w is left of the beak xyz is similar, but uses Q_2 in place of Q_1.

Now suppose that $w \in B$ and that w is a beak peak. Specifically, suppose that $x'wz'$ is a beak of B. Since we have assumed that xyz is an upbeak, we may assume that $x'wz'$ is a downbeak. It suffices to show that in either Q_1 or Q_2, we have that $x' \not< x, z$ and $z' \not< x, z$. Since x' and z' are not beak peaks and are on the same side of x and z since B is a parallel nearly outerplanar graph, the argument of the previous paragraph gives the desired result. (We require that x' and z' be on the same side of x and z so we use the same extension Q_i for both when making the argument.)

Finally, suppose that $w \notin B$. Then there is a path from w to B. Furthermore, because B is a block, the vertex a where the path enters B is independent of the choice of path from w to B. If a is incomparable to x, y, and z, since a is not a beak peak, we apply the argument from our first case and determine that $a \not< x$ and $a \not< z$ in one of Q_1 and Q_2. This implies that w and y remain incomparable in one of the Q_i as desired. If a is comparable to one of x and z, then since w is incomparable to x, y, and z in Q, there is no way for it to become comparable to y in either Q_1 or Q_2. □

With these pieces in place, it is now straightforward to complete the proof of the theorem. Since both Q_1 and Q_2 are outerplanar, $\dim(Q_1) \leq 4$ and $\dim(Q_2) \leq 4$. Let $\{L_5, \ldots, L_8\}$ and $\{L_9, \ldots, L_{12}\}$ be their realizers respectively. Because of the way we constructed Q_1 and Q_2 and the linear extensions $L_0, \ldots, L_4$, it is easy to see that $\{L_0, \ldots, L_{12}\}$ is a realizer of Q. □

Theorem 1.3 now follows as an immediate corollary from this result by the characterization on Kinnersley and Langston from [7]. We also note that Trotter [12] has subsequently made an observation regarding the relationship between dimension and the block structure of the cover graph, making it possible to drop T_3, T_4, and T_5 from the list of forbidden minors. However, the resulting bound on the dimension is weaker in that case.

5. Standard Examples and Treewidth

A second question posed in [5] remains open.

Question 5.1. Is there a constant d such that if P is a poset with cover graph G and $\text{tw}(G) \leq 2$, then $\dim(P) \leq d$?

The following theorem provides some weak evidence for an affirmative answer to this question, since the theorem implies that if the answer to Question 5.1 is "no", a counterexample cannot be constructed using large standard examples.

Theorem 5.2. *If P is a poset that contains S_5 as a subposet, then the cover graph of P has treewidth at least 3.*

Proof. Since the only forbidden minor for a graph to have treewidth at most 2 is K_4, it will suffice to show that the cover graph of P has a K_4-minor. To do this, we build up a series of internally vertex-disjoint paths in the cover graph which we join to create a K_4-minor. We denote such a path between any two elements x and y by $P(x, y)$. In fact, we restrict our attention to paths where x and y are comparable, and thus the path represents a maximal chain between x and y in P.

Let $\{a_1, \ldots, a_5\}$ and $\{b_1, \ldots, b_5\}$ be elements of the subposet of P isomorphic to S_5 with the standard ordering, that is, $a_i < b_j$ if and only if $i \neq j$. As $a_1 < b_2, b_3$ there exist two maximal chains $P(a_1, b_2)$ and $P(a_1, b_3)$. Since these paths need not be internally disjoint, let c_1 be the maximal element occurring in both $P(a_1, b_2)$ and $P(a_1, b_3)$ and fix three internally disjoint paths $P(a_1, c_1)$, $P(c_1, b_2)$, and $P(c_1, b_3)$. Define c_2 and c_3 analogously and notice that $\{c_1, c_2, c_3\}$ is an antichain in P since a_i is incomparable to b_i. In a dual manner we notice that $P(c_2, b_1)$ and $P(c_3, b_1)$ may not be disjoint. Define d_1 as the minimal element in both chains, and define d_2 and d_3 analogously. Thus the poset P contains four (not necessarily disjoint) antichains $\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}, \{c_1, c_2, c_3\}$, and $\{d_1, d_2, d_3\}$ together with internally disjoint paths $P(a_i, c_i)$ and $P(d_i, b_i)$ for $i \in \{1, 2, 3\}$ and paths $P(c_i, d_j)$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$. See Figure 4. We call the subposet on these elements S.

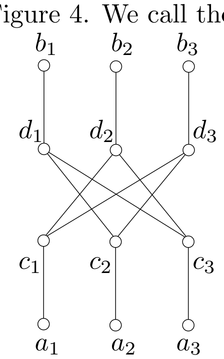


FIGURE 4. The subposet S with vertices internal to chains/paths not shown.

We now note that $P(c_1, d_2), P(d_2, c_3), P(c_3, d_1), P(d_1, c_2), P(c_2, d_3), P(d_3, c_1)$ is a cycle C in the cover graph of P. Thus, if any element of the poset is connected to this cycle by three vertex-disjoint paths, then the cover graph contains a K_4-minor, as desired. Noting that $a_4 < b_1, b_2, b_3$ we now consider the relationship between a_4 and S. Suppose first that a_4 is not less than any element of $\{c_1, c_2, c_3\}$. As every element of $C \setminus \{c_1, c_2, c_3\}$ is less than precisely one element of $\{b_1, b_2, b_3\}$, there are three paths in the cover graph from a_4 to C, creating a K_4-minor. (The paths may not be fully vertex-disjoint, but there is some point a'_4 not on C from which they are vertex-disjoint.)

Therefore, we may assume that a_4 is less than one element of $\{c_1, c_2, c_3\}$, say c_1. By a similar argument, we may assume b_4 is greater than an element of $\{d_1, d_2, d_3\}$. Furthermore, since b_4 is incomparable to a_4 while d_2 and d_3 are comparable to c_1, our assumption that $a_4 < c_1$ forces d_1 to be the d_i that is less than b_4. Thus, there is a vertex β_4 on $P(d_1, b_1)$ such that $\beta_4 < b_4$ and a vertex α'_4 on $P(a_1, c_1)$ such that $a_4 < \alpha'_4$. Since $a_4 < b_1$ and a_4 is incomparable to b_4, there is some element α_4 on $P(d_1, b_1)$ with $\alpha_4 > \beta_4$ and $a_4 < \alpha_4$. Similarly, there is an element β'_4 on $P(a_1, c_1)$ with $\beta'_4 < \alpha'_4$ and $\beta'_4 < b_4$. See Figure 5 for an illustration of the relationship between these points. In a similar manner, we can find a $j \in \{1, 2, 3\}$ and elements β_5 on $P(d_j, b_j)$ and β'_5 on $P(a_j, c_j)$ such that $\beta_5, \beta'_5 < b_5$. There are also elements $\alpha_5, \alpha'_5 > a_5$ such that $\alpha_5 > \beta_5$ on $P(d_j, b_j)$ and $\alpha'_5 > \beta'_5$ on $P(a_j, c_j)$.

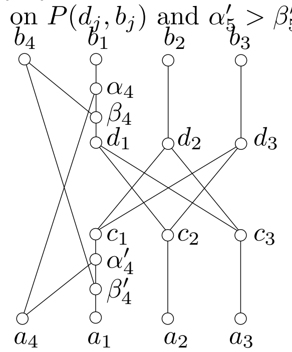


FIGURE 5. Expanding S by adding $a_4, b_4, \alpha_4, \beta_4, \alpha'_4, \beta'_4$.

We consider the cases where $j \neq 1$ and $j = 1$ separately. For the former, suppose without loss of generality that $j = 3$. In this case, the following paths form a K_4-minor with corners[1] $\{c_1, d_1, c_3, d_3\}$:

- $P(d_1, c_2)P(c_2, d_3)$,
- $P(d_3, \beta_5)P(\beta_5, b_5)P(b_5, \beta'_5)P(\beta'_5, c_3)$,
- $P(c_3, d_2)P(d_2, c_1)$,
- $P(c_1, \alpha'_4)P(\alpha'_4, a_4)P(a_4, \alpha_4)P(\alpha_4, d_1)$,
- $P(d_1, c_3)$, and
- $P(d_3, c_1)$.

The most delicate part of our argument remains in the case where $j = 1$. We consider now the paths that enter $P(d_1, b_1)$. Specifically, we examine the relationships between $P(a_4, \alpha_4), P(a_5, \alpha_5), P(b_4, \beta_4)$, and $P(b_5, \beta_5)$. The paths entering $P(a_1, c_1)$ featuring the α'_i and β'_i will interact identically by duality. It is clear that $P(a_4, \alpha_4)$ and $P(b_4, \beta_4)$ do not cross, as otherwise $a_4 < b_4$. Suppose then that $P(a_4, \alpha_4)$ and $P(b_5, \beta_5)$ cross at some point x, while $P(a_5, \alpha_5)$ and $P(b_4, \beta_4)$ do not cross. Furthermore, if the paths $P(a_4, \alpha_4)$ and $P(b_5, \beta_5)$ cross more than once, we will assume that x is the minimal such crossing (in terms of the poset).

Now consider the rerouting of the path $P(d_1, b_1)$ through x with appropriate choices of new vertices $\hat{\alpha}_4, \hat{\alpha}_5, \hat{\beta}_4$, and $\hat{\beta}_5$. The new paths $P(a_4, \hat{\alpha}_4)$ and $P(b_5, \hat{\beta}_5)$ are internally disjoint by construction, and it is straightforward to construct paths $P(a_5, \hat{\alpha}_5)$ and $P(b_4, \hat{\beta}_4)$ that are disjoint unless $\beta_5 < \alpha_5 \leq \beta_4 < \alpha_4$. In this case, consider the cycle formed by $P(\beta_5, x), P(x, \alpha_4)$, and $P(\beta_5, \alpha_4)$. (See Figure 6.) Observe that there are three distinct paths $P(x, a_4), P(\beta_5, d_1)$, and $P(\alpha_5, a_5)$ em-

[1]We use "corners" here to mean that the given vertices are in distinct branch sets of the K_4-minor without fully specifying the branch sets.

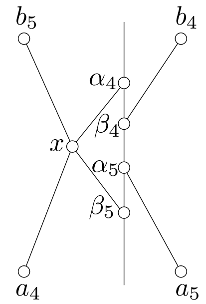


FIGURE 6. Rerouting $P(d_1, b_1)$ via x.

anating from the cycle. Since a_4, a_5, and d_1 all connect to the path $P(a_1, c_1)$, we obtain a K_4-minor by contracting these vertices to a single vertex.

Now consider the case where, in addition, $P(a_5, \alpha_5)$ and $P(b_4, \beta_4)$ cross at some point y, again choosing y as the minimal intersection point. Since $\beta_5 < x < \alpha_4$, $\beta_4 < y < \alpha_5$, $\beta_4 < \alpha_4$, and $\beta_5 < \alpha_5$, we have that $\{\beta_4, \beta_5\} < \{\alpha_4, \alpha_5\}$. Since a_i is incomparable to b_i in the poset, we must have that x and y are incomparable as well. This implies that any intersection between $P(x, \beta_5)$ and $P(y, \beta_4)$ occurs at a point less than both x and y on these paths. Similarly, any intersection between $P(x, \alpha_4)$ and $P(y, \alpha_5)$ must be greater than both x and y. It is then easy to see that there is a $(K_4 - e)$-minor with corners $x, y, \{\beta_4, \beta_5\}$, and $\{\alpha_4, \alpha_5\}$, possibly adding intersection points between $P(x, \beta_5)$ and $P(y, \beta_4)$ to $\{\beta_4, \beta_5\}$ and intersection points between $P(x, \alpha_4)$ and $P(y, \alpha_5)$ to $\{\alpha_4, \alpha_5\}$. The missing connection to complete a K_4-minor is between x and y. However, notice that x and y are connected by a path through a_4, a_5 and $P(a_1, c_1)$, giving the needed connection to complete the minor.

We are now able to make a fairly strong assumption about the pairwise intersections of $P(a_4, \alpha_4), P(a_5, \alpha_5), P(b_4, \beta_4)$, and $P(b_5, \beta_5)$. Of the six possible crossings, the only two that can occur are $P(a_4, \alpha_4)$ with $P(a_5, \alpha_5)$ and $P(b_4, \beta_4)$ with $P(b_5, \beta_5)$. Furthermore, these intersections imply that $\alpha_4 = \alpha_5$ or $\beta_4 = \beta_5$, respectively.

Having established these intersection limitations (and the corresponding ones for the α_i' and β_i'), we consider the graph formed by contracting each of $P(a_i, \alpha_i)$, $P(a_i, \alpha_i')$, $P(b_i, \beta_i)$, and $P(b_i, \beta_i')$ for $i = 4, 5$ to a single edge. In fact, we go further and contract all the edges we can while ensuring that the α_i, β_i, α_i', β_i', a_i, and b_i are not identified. Since it is possible for some of these vertices to have been equal at the outset, we are then left with a graph with at most 12 vertices. (We refer to the vertices as having labels to allow that, for example, α_4 and α_5 may refer to the same vertex.) The resulting graph is a path which has two blocks of vertices, one consisting of vertices with labels $V = \{\alpha_4, \alpha_5, \beta_4, \beta_5\}$ and one consisting of vertices with labels $V' = \{\alpha_4', \alpha_5', \beta_4', \beta_5'\}$, together with a collection of 4 paths of length 2 (via the a_i and b_i) connecting vertices in the two blocks.

We now show that in all but one case to be described later, this graph has a K_4-minor. Let M' be the labels of the maximum vertex (with respect to the poset) of the vertices with labels in V' and similarly define M as the labels of the minimal vertex of the vertices with labels in V. We claim that $M \in \{\{\beta_4\}, \{\beta_5\}, \{\beta_4, \beta_5\}\}$ and $M' \in \{\{\alpha_4'\}, \{\alpha_5'\}, \{\alpha_4', \alpha_5'\}\}$. This is because M and M' each correspond to a single vertex, so they cannot contain two labels with the same subscript and

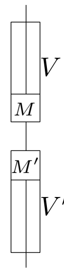


FIGURE 7. Relation of blocks of labels.

must respect the ordering on elements with the same subscript. We now construct the K_4-minor using M, $V\backslash M$, M', and $V'\backslash M'$ as the corners. Figure 7 makes clear that (with appropriate contractions), $V\backslash M, M, M', V'\backslash M'$ is a path of length 3. Therefore, it suffices to show that there are connections between (1) M and $V'\backslash M'$, (2) M' and $V\backslash M$, and (3) $V\backslash M$ and $V'\backslash M'$. Observe that two sets S and S' are connected through $\{a_4, a_5, b_4, b_5\}$ if, ignoring primes, $S \cap S'$ is non-empty. For instance, if $\beta_4 \in S$ and $\beta_4' \in S'$, then the path via b_4 provides the desired connection. This characterization immediately implies there is a connection between M and $V'\backslash M'$ and between M' and $V\backslash M$ for all possible choices of M and M'. Furthermore, the only case where there isn't a connection between $V\backslash M$ and $V'\backslash M'$ is when $M = \{\beta_4, \beta_5\}$ and $M' = \{\alpha_4', \alpha_5'\}$.

In this case, to avoid a K_4-minor the poset must contain the paths depicted in Figure 8. However, in this case there is a path between a_4 and b_5 since $a_4 < b_5$ in the poset. The fact that a_i and b_i are incomparable in the poset restricts the potential intersection points of the path from a_4 to b_5 in a way that makes it straightforward to verify there is a K_4-minor with corners $\{a_4, b_5, \alpha_4', \beta_5\}$. □

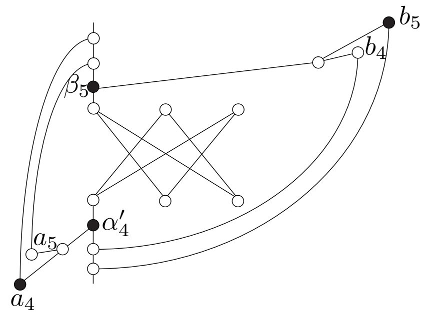


FIGURE 8. There must also be a path from b_5 to a_4.

We conclude this section by observing that for any $x \in S_5$, there is a poset containing $S_5\backslash\{x\}$ and having a cover graph of treewidth 2. We show an example in Figure 9, with the poset on the left and a redrawing of the cover graph on the right. (Since the graph is clearly K_4-minor-free, it has treewidth at most 2.) This implies that Theorem 5.2 is best possible.

REFERENCES

[1] BARÁT, J., HAJNAL, P., LIN, Y., AND YANG, A. On the structure of graphs with path-width at most two. *Studia Sci. Math. Hungar. 49*, 2 (2012), 211–222.

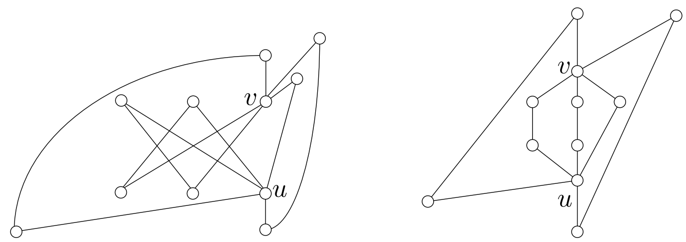


FIGURE 9. A poset containing $S_5 \setminus \{x\}$ (left) with cover graph (right) redrawn to make clear it has treewidth 2.

[2] FELSNER, S., LI, C. M., AND TROTTER, W. T. Adjacency posets of planar graphs. *Discrete Math. 310*, 5 (2010), 1097–1104.

[3] FELSNER, S., TROTTER, W. T., AND WIECHERT, V. Posets with outerplanar cover graphs. To appear in *Graphs and Combinatorics* (2013).

[4] HALIN, R. *S*-functions for graphs. *J. Geometry 8*, 1-2 (1976), 171–186.

[5] JORET, G., MICEK, P., MILANS, K. G., TROTTER, W. T., WALCZAK, B., AND WANG, R. Tree-width and dimension. arXiv:1301.5271v1, 2013.

[6] KELLY, D. On the dimension of partially ordered sets. *Discrete Math. 35* (1981), 135–156.

[7] KINNERSLEY, N. G., AND LANGSTON, M. A. Obstruction set isolation for the gate matrix layout problem. *Discrete Appl. Math. 54*, 2-3 (1994), 169–213. Efficient algorithms and partial *k*-trees.

[8] ROBERTSON, N., AND SEYMOUR, P. D. Graph minors. III. Planar tree-width. *J. Combin. Theory Ser. B 36*, 1 (1984), 49–64.

[9] ROBERTSON, N., AND SEYMOUR, P. D. Graph minors. XX. Wagner's conjecture. *J. Combin. Theory Ser. B 92*, 2 (2004), 325–357.

[10] STREIB, N., AND TROTTER, W. T. Dimension and height for posets with planar cover graphs. Submitted.

[11] TROTTER, W. T. *Combinatorics and partially ordered sets: Dimension theory.* Johns Hopkins Series in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 1992.

[12] TROTTER, W. T. Personal communication. 2013.

[13] TROTTER, JR., W. T., AND MOORE, JR., J. I. The dimension of planar posets. *J. Combinatorial Theory Ser. B 22*, 1 (1977), 54–67.

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