1. (5 points) Given that $\int \sin^n(t) = -\frac{\sin^{n-1}(t)}{n}\cos(t) + \frac{n-1}{n}\int \sin^{n-2}(t)dt$ find $\int_0^\pi \sin^3(3t)dt$.

Solution: First note that letting $w = 3t$, then $dw = 3dt$.
Thus $\int_0^\pi \sin^3(3t)dt = \frac{1}{3}\int_0^{3\pi} \sin^3(w)dw$. Now

$$\begin{aligned}\int \sin^3(w)dw &= -\frac{\sin^2(w)}{3}\cos(w) + \frac{2}{3}\int \sin(w)dw \\ &= -\frac{1}{3}\sin^2(w)\cos(w) + \frac{2}{3}(-\cos(w)) + C.\end{aligned}$$

Thus $\int_0^{3\pi} \sin(w)dw = -\frac{1}{3}\sin^2(3\pi)\cos(3\pi) - \frac{2}{3}\cos(3\pi) + \frac{1}{3}\sin^2(0)\cos(0) + \frac{2}{3}\cos(0) = \frac{2}{3} + \frac{2}{3} = -\frac{4}{3}$. Thus $\int_0^\pi \sin^3(3t)dt = \frac{4}{9}$.

2. (5 points) Find $\frac{d}{dt}\int_{t^2}^{\sin(t)} e^{-x^2}dx$.

Solution: By the fundamental theorem of calculus, e^{-x^2} has some anti-derivative $F(x)$. Thus $\int_{t^2}^{\sin(t)} e^{-x^2}dx = F(\sin(t)) - F(t^2)$. Taking the derivative with respect to t, we get

$$F'(\sin(t))\cos(t) - F'(t^2)2t = \cos(t)e^{-\sin^2(t)} - 2te^{-t^4}.$$

3. (5 points) Find $\int_0^1 e^{2t}\cos(2t)dt$.

Solution: We first find $\int e^{2t}\cos(2t)dt$. Setting $u = \cos(2t)$ and $v' = e^{2t}$ we have that $u' = -2\sin(2t)$ and $v = \frac{1}{2}e^{2t}$. Thus

$$\int e^{2t}\cos(2t)dt = \frac{1}{2}e^{2t}\cos(2t) - \int -e^{2t}\sin(2t)dt.$$

Performing integration by parts again with $u = \sin(2t)$ and $v' = e^{2t}$, we have $u' = 2\cos(2t)$ and $v = \frac{1}{2}e^{2t}$. Thus

$$\int e^{2t}\sin(2t)dt = \frac{1}{2}e^{2t}\sin(2t) - \int \cos(2t)e^{2t}dt.$$

Combining with the previous integration by parts we have

$$\int e^{2t}\cos(2t)dt = \frac{1}{2}e^{2t}\cos(2t) + \frac{1}{2}e^{2t}\sin(2t) - \int e^{2t}\cos(2t)dt.$$

Solving for the integral we get that

$$\int e^{2t}\cos(2t)dt = \frac{e^{2t}}{4}\left(\cos(2t) + \sin(2t) + C\right).$$

Thus $\int_0^1 e^{2t}\cos(2t)dt = \frac{e^2}{4}\left(\cos(2) + \sin(2)\right) - \frac{1}{4}$.

4. (5 points) Approximate $\int_{-1}^{1} ax^2+bx+c\,dx$ using two intervals and either the Trapezoidal Rule or the Midpoint Rule. Compare your result with the actual answer, is it an over estimate or underestimate? (Hint: This will depend on the value of a.)

Solution: First

$$\begin{aligned}\int_{-1}^{1} ax^2+bx+cdx &= \left[\frac{a}{3}x^3+\frac{b}{2}x^2+cx\right]_{x=-1}^{1}\\ &= \frac{a}{3}+\frac{b}{2}+c-\left(-\frac{a}{3}+\frac{b}{2}-c\right)\\ &= \frac{2a}{3}+2c.\end{aligned}$$

Using the trapeziodal rule with two intervals the estimate would be

$$\begin{aligned}\frac{f(-1)+f(0)}{2}(0-(-1))+\frac{f(0)+f(1)}{2}(1-0) &= \frac{1}{2}f(-1)+f(0)+\frac{1}{2}f(1)\\ &= \frac{1}{2}(a-b+c)+c+\frac{1}{2}(a+b+c)=a+2c.\end{aligned}$$

Thus the difference between the trapeziodal rule and the actual answer is $a+2c-(\frac{2}{3}a+c)=\frac{1}{3}a$. Thus the trapezoidal rule is an overestimate if $a>0$, an underestimate if $a<0$ and exact if $a=0$.

5. (5 points) Find

$$\int \frac{3x^3-9x^2+8x+12}{(x^2+4)(x-2)^2}dx.$$

Solution: We first do the partial fraction decomposition. That is, we find A,B,C,D so that

$$\frac{3x^3-9x^2+8x+12}{(x^2+4)(x-2)^2}=\frac{A}{(x-2)}+\frac{B}{(x-2)^2}+\frac{Cx+D}{x^2+4}.$$

Then we get

$$\begin{aligned}3x^3-9x^2+8x+12 &= A(x-2)(x^2+4)+B(x^2+4)+(CX+D)(x-2)^2\\ &= (A+C)x^3+(-2A+B-2C+D)x^2+(4A+4C-4D)x+(-8A+4B+4D).\end{aligned}$$

Thus we get

$$\begin{aligned}A+C&=3\\ -2A+B-4C+D&=-9\\ 4A+4C-4D&=8\\ -8A+4B+4D&=12\end{aligned}$$

which has solution $A = 0$, $B = 2$, $C = 3$, and $D = 1$. Thus

$$\frac{3x^3 - 9x^2 + 8x + 12}{(x^2+4)(x-2)^2} = \frac{2}{(x-2)^2} + \frac{3x}{x^2+4} + \frac{1}{x^2+4}$$

and

$$\int \frac{3x^3 - 9x^2 + 8x + 12}{(x^2+4)(x-2)^2} dx = \frac{-2}{(x-2)} + \frac{3}{2}\ln(x^2+4) + \frac{1}{2}\arctan(\frac{x}{2}) + C.$$

6. (5 points) Find $\int \frac{4}{y^2\sqrt{9-y^2}}dy$. (Hint: It may be helfpul to use $\int \csc^2(\theta)d\theta = -\cot(\theta) + C$ or $\int \sec^2(\theta)d\theta = \tan(\theta) + C$.)

Solution: Since we have a difference of squares, we use trig. substitution with a triangle that has hypotenuse 3 and one leg y. We choose to put θ so that $\sin(\theta) = \frac{y}{3}$. Thus $\cos(\theta)d\theta = \frac{1}{3}dy$ and $\cos(\theta) = \frac{\sqrt{9-y^2}}{3}$. Using this substitution we have

$$\int \frac{4}{y^2\sqrt{9-y^2}}dy = \int 4\frac{1}{y^2}\frac{1}{\sqrt{9-y^2}}dy = \int 4\frac{1}{9\sin^2(\theta)}\frac{1}{3\cos(\theta)}3\cos(\theta)d\theta = \frac{4}{9}\int \csc^2(\theta)d\theta.$$

Using the provided integral, this is $-\frac{4}{9}\cot(\theta) + C = -\frac{4}{9}\cot(\arcsin(\frac{y}{3})) + C = -\frac{4}{9}\frac{\sqrt{9-y^2}}{y}$.

7. (5 points) Show that $\int_3^5 2\ln(s^2+s)e^{-s^2}ds = \int_9^{25} \frac{\ln(t+\sqrt{t})}{\sqrt{t}}e^{-t}dt$.

Solution: Comparing the two integrals we see that one has e^{-s^2} and one has a e^{-t}, so we will try the substitution $s^2 = t$. Thus $2sds = dt$ and if $t = 9$ then $s = 3$ and if $t = 25$ then $s = 5$. Thus

$$\int_9^{25} \frac{\ln(t+\sqrt{t})}{\sqrt{t}}e^{-t}dt = \int_3^5 \frac{\ln(s^2+\sqrt{s^2})}{\sqrt{s^2}}e^{-s^2}2sds = \int_3^5 2\ln(s^2+s)e^{-s^2}ds$$

and the two integrals are equal.