1. (5 points) Show that $y = \cos(t) + t^2 + 4$ satisfies the differential equation $\frac{d^2y}{dt^2} + y = 6 + t^2$.

Solution: First note that $\frac{d^2y}{dt^2} = \frac{d}{dt}(-\sin(t)+2t) = -\cos(t)+2$. Thus $\frac{d^2y}{dt^2}+y = (-\cos(t)+2) + (\cos(t) + t^2 + 4) = 6 + t^2$. Thus $y = \cos(t) + t^2 + 4$, satisfies the differential equation $\frac{d^2y}{dt^2} + y = 6 + t^2$.

2. (5 points) Does the integral $\int_1^\infty \frac{\sin^2(x)}{x^2}dx$ converge or diverge? If it converges, provide an upper bound for its value.

Solution: First we note that $0 \le \sin^2(x) \le 1$ so $\frac{\sin^2(x)}{x^2} \le \frac{1}{x^2}$. Since

$$\begin{aligned}
\int_1^\infty \frac{1}{x^2}dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^2}dx \\
&= \lim_{b\to\infty} \left[\frac{-1}{x}\right]_{x=1}^b \\
&= \lim_{b\to\infty} \frac{-1}{b} + 1 \\
&= 1.
\end{aligned}$$

Thus $\int_1^\infty \frac{\sin^2(x)}{x^2}dx$ converges and is at most 1.

3. (5 points) Does the integral $\int_0^{\frac{\pi}{2}} \frac{e^{-\tan(x)}}{\cos^2(x)}dx$ converge? If so, find its value. (Recall that $\sec^2(x) = \frac{1}{\cos^2(x)}$.)

Solution: Since $\cos(\frac{\pi}{2}) = 0$ this is an improper integral. We also note that letting $u = \tan(x)$ and $du = sec^2(x)dx$, $\int e^{-\tan(x)}\cos^2(x)dx = \int e^{-u}du = -e^{-u} + C = -e^{-\tan(x)}$. Thus

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} e^{-\tan(x)}\sec^2(x)dx &= \lim_{b\to\frac{\pi}{2}^-} \int_0^b e^{-\tan(x)}\sec^2(x)dx \\
&= \lim_{b\to\frac{\pi}{2}^-} \left[-e^{-\tan(x)}\right]_{x=0}^b \\
&= \lim_{b\to\frac{\pi}{2}^-} -e^{-\tan(b)} + e^{-\tan(0)} \\
&= \lim_{b\to\frac{\pi}{2}^-} -e^{-\tan(b)} + 1 \\
&= 1.
\end{aligned}$$

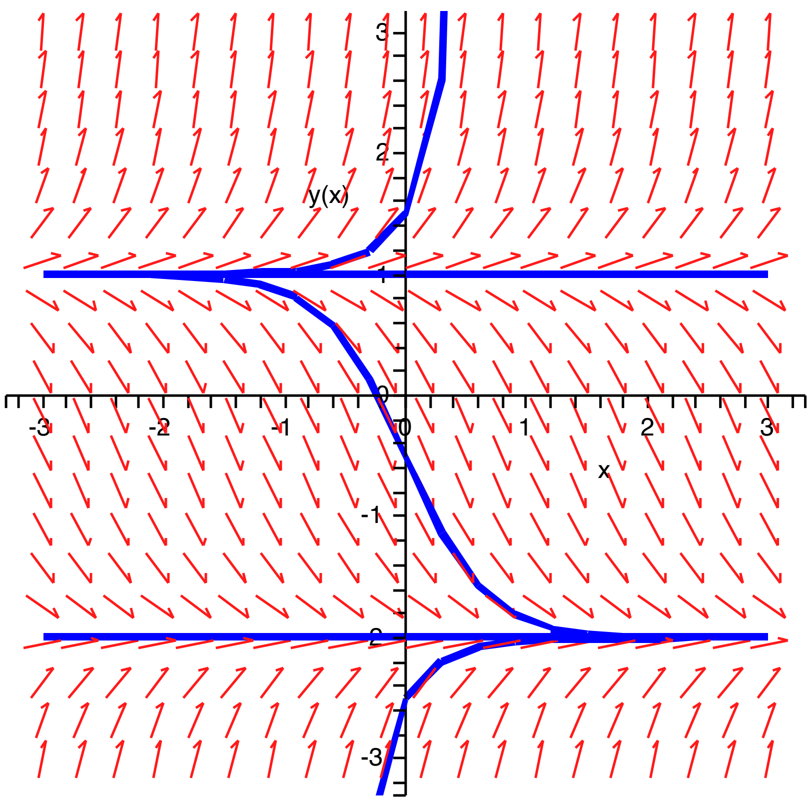
4. (5 points) Consider the region bounded by $y = 3x^2$, $y = 0$ and $x = 2$. Find the volume of the object formed by rotating this region about the x-axis.

Solution: The cross sections are cylinders with radius $3x^2$ and height Δx, and thus the volume is

$$\begin{aligned}\int_0^2 \pi(3x^2)^2 dx &= \pi \int_0^2 9x^4 dx \\ &= \pi \left[\frac{9}{5}x^5\right]_{x=0}^2 \\ &= \pi \frac{9}{5} 2^5 \\ &= \frac{288}{5}\pi.\end{aligned}$$

5. (5 points) Plot 5 representative solutions for the differential equation represented by the following slope field. (Hint: two of these should be equilibrium solutions)

Solution:



6. (5 points) Find the solution curve through $(-2, 3)$ for the differential equation $\frac{dy}{dt} = y(t+1)$.

Solution: We first separate variables, yielding $\frac{dy}{y} = (t+1)dt$. Integrating both sides we get $\ln|y| = \frac{t^2}{2} + t + C$. Solving for y we get $|y| = e^{\frac{t^2}{2}+t+C} = e^C e^{\frac{t^2}{2}+t}$ and then $y = \pm e^C e^{\frac{t^2}{2}+t}$. We then replace $\pm e^{-C}$ with the arbitrary constant B. So we have $y = Be^{\frac{t^2}{2}+t}$. The solution curve through $(-2, 3)$ satisfies that $3 = y(-2) = Be^{\frac{(-2)^2}{2}+(-2)} = Be^0 = B$, thus $B = 3$ and thus the solution curve through $(-2, 3)$ is $y(t) = 3^{\frac{t^2}{2}+t}$.

7. (5 points) Todd the Tea salesman is selling iced tea at a street fair from a 100 gallon container. Unfortunately for Todd, his container is full of "Southern-style" sweet tea which has half a pound of sugar per gallon and the crowd wants lightly sweetened tea, containing a pound of sugar per 10 gallons. Assuming Todd sells his tea at a rate of 10 gallons an hour and it is replenished by a mixture with a pound of sugar per 20 gallons, how long until Todd's tea is not too sweet for his customers? (Assume that the tea in the 100 gallon container is well mixed.)

Solution: Let S be the pounds of sugar in the 100 gallon tank. Since it the container starts at half a pound of sugar per gallon, there are 50 pounds of sugar initially in the container. The goal is to reach a concentration of a pound of sugar per 10 gallons, or 10 total gallons in the container. Sugar flows out of the container at a rate of $\frac{S(t)}{100}10$ pounds an hour. Sugar flows into the container at a rate of $\frac{1}{20}10$ pounds an hour. Thus the differential equation that is satisfied by the sugar is $S(0) = 50$ and $\frac{dS}{dt} = \frac{1}{2} - \frac{S}{10} = \frac{1}{10}(()5 - S)$.

Thus, by separating variables we have $\frac{dS}{5-S} = \frac{dt}{10}$ and $-\ln|5 - S| = \frac{t}{10} + C$, Solving for S, we get $S = 5 + Ce^{-\frac{t}{10}}$. Since $S(0) = 50$, we have $C = 45$. We want to know at which t we have $10 = S(t) = 5 + 45e^{-\frac{t}{10}}$. Now solving for t we have $\ln(\frac{1}{9}) = -\frac{t}{10}$, or $10\ln(9) = t$. So after $10\ln(9)$ hours Todd will have a sufficiently unsweet tea.

8. (5 points) You have an opportunity to lease a franchise of an up and coming new store for 10 years for \$100,000. The income stream is expected to be \$15,000 per year in those 10 years. If the interest rate is expected to hold steady at 10% compounded continuously for the next 10 years, what should you do?

Solution: We have to discount the value of the income stream over the 10 years to find the current value of the money. That is

$$\begin{aligned}\int_0^{10} 15000e^{-\frac{t}{10}}dt &= 15000\int_0^{10} e^{-\frac{t}{10}}dt \\ &= 15000\left[10e^{-\frac{t}{10}}\right]_{t=0}^{10} \\ &= 150000(-e^{-1}+1) \\ &\approx 94818.08\end{aligned}$$

Thus investing in the franchise is not a good choice.