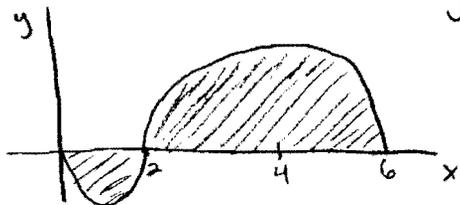


## MATH 20B HW1

- Section 5.2: 13, 51, 58, 80
- Section 5.3: 11, 20, 35, 41, 46
- Section 5.4: 5, 11, 18, 21, 31
- Section 5.5: 8, 13
- Section 5.6: 13, 17, 24, 26, 34, 40, 47, 60, 83, 84, 91
- Section 6.1: 3, 16, 18, 30, 37, 42, 50

5.2.13

Evaluate the integrals for  $f(x)$  shown in Figure 14.

The two parts of the graph are semicircles.

$$\begin{aligned} (a) \int_0^2 f(x) dx &= -\frac{1}{2} \pi (1)^2 \\ &= \boxed{-\frac{\pi}{2}} \end{aligned}$$

$$\begin{aligned} (b) \int_0^6 f(x) dx &= \int_0^2 f(x) dx + \int_2^6 f(x) dx \\ &= -\frac{\pi}{2} + \frac{1}{2} \pi (2)^2 \\ &= -\frac{\pi}{2} + 2\pi \\ &= \boxed{\frac{3\pi}{2}} \end{aligned}$$

$$\begin{aligned} (c) \int_1^4 f(x) dx &= \int_1^2 f(x) dx + \int_2^4 f(x) dx \\ &= -\frac{1}{4} \pi (1)^2 + \frac{1}{4} \pi (2)^2 \\ &= -\frac{\pi}{4} + \pi \\ &= \boxed{\frac{3\pi}{4}} \end{aligned}$$

$$\begin{aligned} (d) \int_1^6 |f(x)| dx &= \int_1^2 |f(x)| dx + \int_2^6 |f(x)| dx \\ &= \frac{\pi}{4} + 2\pi = \boxed{\frac{9\pi}{4}} \end{aligned}$$

5.2.51

Calculate the integral, assuming that

$$\int_0^5 f(x) dx = 5, \quad \int_0^5 g(x) dx = 12$$

$$\begin{aligned} \int_0^5 [f(x) + 4g(x)] dx &= \int_0^5 f(x) dx + \int_0^5 4g(x) dx \\ &= \int_0^5 f(x) dx + 4 \int_0^5 g(x) dx \\ &= 5 + 4 \cdot 12 \\ &= \boxed{53} \end{aligned}$$

5.2.58

Calculate the integral, assuming that

$$\int_0^1 f(x) dx = 1, \quad \int_0^2 f(x) dx = 4, \quad \int_1^4 f(x) dx = 7$$

$$\begin{aligned} \int_2^4 f(x) dx &= \int_1^4 f(x) dx - \int_1^2 f(x) dx \\ &= \int_1^4 f(x) dx - \left[ \int_0^2 f(x) dx - \int_0^1 f(x) dx \right] \\ &= 7 - [4 - 1] \\ &= 7 - 3 \\ &= \boxed{4} \end{aligned}$$

5.2.80 Find upper and lower bounds for  $\int_0^1 \frac{dx}{\sqrt{x^2+4}}$ .

We might like to try something with left- and right-approximations, but we don't have this framework yet.

Graphing  $f(x) = \frac{1}{\sqrt{x^2+4}}$  shows us that  $f$  is decreasing on  $[0, 1]$  (we could also show that  $f'(x) < 0$  on this interval), so that on  $[0, 1]$

$$f(1) \leq f(x) \leq f(0)$$
$$\frac{1}{\sqrt{5}} \leq f(x) \leq \frac{1}{2}$$

So we can apply the Comparison Test to obtain

$$\frac{1}{\sqrt{5}} (1-0) \leq \int_0^1 f(x) dx \leq \frac{1}{2} (1-0)$$

$$\boxed{\frac{1}{\sqrt{5}} \leq \int_0^1 f(x) dx \leq \frac{1}{2}}$$

which gives us our needed bounds.

5.3.11 In Exercises 11, 20, and 35, evaluate the integral using FTC I.

$$\begin{aligned} \int_{-2}^0 (3x - 2e^x) dx &= \left[ \frac{3x^2}{2} - 2e^x \right]_{-2}^0 \\ &= (0 - 2) - (6 - 2e^{-2}) \\ &= -2 - 6 + 2e^{-2} \\ &= \boxed{\frac{2}{e^2} - 8} \end{aligned}$$

5.3.20

$$\begin{aligned}
 \int_1^2 (x^2 - x^{-2}) dx &= \left[ \frac{x^3}{3} + \frac{1}{x} \right]_1^2 \\
 &= \left( \frac{8}{3} + \frac{1}{2} \right) - \left( \frac{1}{3} + 1 \right) \\
 &= \frac{19}{6} - \frac{4}{3} \\
 &= \boxed{\frac{11}{6}}
 \end{aligned}$$

5.3.35

$$\begin{aligned}
 \int_{\pi/4}^{3\pi/4} \sin \theta d\theta &= -\cos \theta \Big|_{\pi/4}^{3\pi/4} \\
 &= -\left( \cos \frac{3\pi}{4} - \cos \frac{\pi}{4} \right) \\
 &= -\left( -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) \\
 &= -(-\sqrt{2}) \\
 &= \boxed{\sqrt{2}}
 \end{aligned}$$

5.3.41

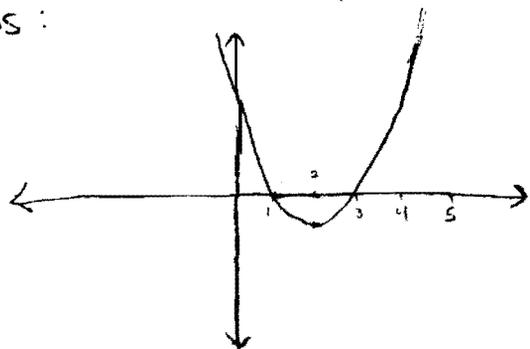
In Exercises 41 and 46, write the integral as a sum of integrals without absolute values and evaluate.

$$\begin{aligned}
 \int_{-2}^1 |x| dx &= \int_{-2}^0 -x dx + \int_0^1 x dx \\
 &= \left[ -\frac{x^2}{2} \right]_{-2}^0 + \left[ \frac{x^2}{2} \right]_0^1 \\
 &= -\left( 0 - \frac{(-2)^2}{2} \right) + \left( \frac{1}{2} - \frac{0}{2} \right) \\
 &= -(-2) + \frac{1}{2} \\
 &= \boxed{\frac{5}{2}}
 \end{aligned}$$

5.3.46

$$\int_0^5 |x^2 - 4x + 3| dx$$

Let's graph this. First note that  $x^2 - 4x + 3 = (x-1)(x-3)$ , so its graph is as follows:



This means that we split the integral into three parts:

$$\int_0^5 |x^2 - 4x + 3| dx = \int_0^1 (x^2 - 4x + 3) dx + \int_1^3 -(x^2 - 4x + 3) dx + \int_3^5 (x^2 - 4x + 3) dx$$

$$= \left[ \frac{x^3}{3} - 2x^2 + 3x \right]_0^1 - \left[ \frac{x^3}{3} - 2x^2 + 3x \right]_1^3$$

$$+ \left[ \frac{x^3}{3} - 2x^2 + 3x \right]_3^5$$

$$= \left[ \left( \frac{1}{3} - 2 + 3 \right) - (0) \right] - \left[ (9 - 18 + 9) - \left( \frac{1}{3} - 2 + 3 \right) \right] + \left[ \left( \frac{125}{3} - 50 + 15 \right) - (9 - 18 + 9) \right]$$

$$= \left[ \frac{4}{3} - 0 \right] - \left[ 0 - \frac{4}{3} \right] + \left[ \frac{20}{3} - 0 \right]$$

$$= \boxed{\frac{28}{3}}$$

5.4.5

Find  $G(1)$ ,  $G'(0)$ , and  $G'(\pi/4)$ , where

$$G(x) = \int_1^x \tan t \, dt$$

First note that

$$G(1) = \int_1^1 \tan t \, dt = \boxed{0}$$

Next observe that FTC II gives

$$G'(x) = \tan x$$

so that

$$G'(0) = \tan 0 = \boxed{0}$$

$$G'\left(\frac{\pi}{4}\right) = \tan \frac{\pi}{4} = \boxed{1}$$

5.4.11

Find a formula for the function represented by the integral.

$$\int_x^5 e^t \, dt$$

By noting that  $e^t$  is an antiderivative of  $e^t$ , we can integrate by FTC I to get

$$\int_x^5 e^t \, dt = e^t \Big|_x^5 = \boxed{e^5 - e^x}$$

5.4.18

Express the antiderivative  $F(x)$  of  $f(x)$  satisfying the given initial condition as an integral.

$$f(x) = e^{-x^2} \quad F(4) = 0$$

This means that we need to apply FTC II to get

$$F(x) = \int_a^x f(t) \, dt = \int_a^x e^{-t^2} \, dt$$

which amounts to finding the particular

value of  $a$ . But if  $F(4) = 0$ , then we can take  $a = 4$  since then

$$F(4) = \int_4^4 e^{-t^2} dt = 0$$

as needed. Thus

$$F(x) = \int_4^x e^{-t^2} dt$$

5.4.21 Calculate the derivative.

$$\frac{d}{dt} \int_{100}^t \cos 5x dx$$

This is a simple application of FTC II:

$$\frac{d}{dt} \int_{100}^t \cos 5x dx = \cos 5t$$

5.4.31 Calculate the derivative.

$$\frac{d}{ds} \int_{-6}^{\cos s} (u^4 - 3u) du$$

Here we need to combine the FTC and the Chain Rule as in Example 4. So set  $x = \cos s$  ( $u$  is already taken) and so we obtain

$$\begin{aligned} \frac{d}{ds} \int_{-6}^{\cos s} (u^4 - 3u) du &= \frac{d}{dx} \int_{-6}^x (u^4 - 3u) du \cdot \frac{dx}{ds} \\ &= (x^4 - 3x) \cdot (-\sin s) \end{aligned}$$

Plugging  $\cos s$  back in for  $x$ , we have

$$\begin{aligned} \frac{d}{ds} \int_{-6}^{\cos s} (u^4 - 3u) du &= (\cos^4 s - 3\cos s)(-\sin s) \\ &= \boxed{\sin s (3\cos s - \cos^4 s)} \end{aligned}$$

5.5.8

A projectile is released with initial (vertical) velocity  $100 \text{ m/s}$ . Use the formula  $v(t) = 100 - 9.8t$  for velocity to determine the distance traveled during the first  $15 \text{ s}$ .

The key word here is "distance", which means we need to integrate the absolute value of velocity. The interval will be  $[0, 15]$  since we are considering the first  $15 \text{ s}$ :

$$\begin{aligned} \text{distance traveled} &= \int_0^{15} |v(t)| dt \\ &= \int_0^{15} |100 - 9.8t| dt \end{aligned}$$

Note that  $100 - 9.8t > 0$  for  $t < 500/49$  and  $100 - 9.8t < 0$  for  $t > 500/49$ , so using what we learned in §5.3 we have

$$\begin{aligned} \text{distance traveled} &= \int_0^{500/49} (100 - 9.8t) dt + \int_{500/49}^{15} (9.8t - 100) dt \\ &= \left[ 100t - 4.9t^2 \right]_0^{500/49} + \left[ 4.9t^2 - 100t \right]_{500/49}^{15} \\ &= \left[ \left( \frac{50,000}{49} - \frac{25,000}{49} \right) - (0 - 0) \right] \\ &\quad + \left[ (4.9 \cdot 225 - 1500) - \left( \frac{50,000}{49} - \frac{25,000}{49} \right) \right] \\ &\approx 622.9 \text{ m} \end{aligned}$$

5.5.13

The rate (in liters per minute) at which water drains from a tank is recorded in half-minute intervals. Use the average of the left- and right- approximations to estimate the total amount of water drained during the first 3 minutes.

Our units of time are minutes, so we have

$$\begin{aligned}\text{Left-approximation} &= \frac{3-0}{6} [50 + 48 + 46 + 44 + 42 + 40] \\ &= \frac{1}{2} [270] \\ &= 135 \text{ liters}\end{aligned}$$

$$\begin{aligned}\text{Right-approximation} &= \frac{3-0}{6} [48 + 46 + 44 + 42 + 40 + 38] \\ &= \frac{1}{2} [258] \\ &= 129 \text{ liters}\end{aligned}$$

Thus the average of the left- and right- endpoint approximations is 132 liters.

Note: The average of the left- and right- endpoint approximations is the same as the Trapezoidal Rule, which you might have some familiarity with already and which we shall cover later.

5.6.13

In Exercises 13, 17, 24, and 26, write the integral in terms of  $u$  and  $du$ . Then integrate.

$$\int \frac{x+1}{(x^2+2x)^3} dx, \quad u = x^2 + 2x$$

$$\frac{du}{dx} = 2x+2 \Rightarrow dx = \frac{du}{2x+2}$$

$$= \int \frac{x+1}{(x^2+2x)^3} dx = \int \frac{x+1}{u^3} \cdot \frac{du}{2x+2}$$

$$= \int \frac{x+1}{u^3} \cdot \frac{du}{2(x+1)}$$

$$= \int \frac{du}{2u^3}$$

$$= \frac{u^{-2}}{-2} + C$$

$$= -\frac{1}{2u^2} + C$$

$$= \boxed{-\frac{1}{4(x^2+2x)^2} + C}$$

$$\int x^2 \sqrt{4x-1} dx, \quad u = 4x-1 \Rightarrow x = \frac{u+1}{4}$$

$$\frac{du}{dx} = 4 \Rightarrow dx = \frac{1}{4} du$$

$$\int x^2 \sqrt{4x-1} dx = \int \left(\frac{u+1}{4}\right)^2 \sqrt{u} \cdot \frac{1}{4} du$$

$$= \frac{1}{64} \int (u+1)^2 \sqrt{u} du$$

$$= \frac{1}{64} \int (u^2 + 2u + 1) \sqrt{u} du$$

$$= \frac{1}{64} \int (u^{5/2} + 2u^{3/2} + u^{1/2}) du$$

$$= \frac{1}{64} \left[ \frac{u^{7/2}}{7/2} + \frac{2u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2} \right] + C$$

5.6.17

(Sols. manual  
wrong)

$$= \frac{u^{7/2}}{224} + \frac{u^{5/2}}{80} + \frac{u^{3/2}}{96} + C$$

$$= \frac{(4x-1)^{7/2}}{224} + \frac{(4x-1)^{5/2}}{80} + \frac{(4x-1)^{3/2}}{96} + C$$

5.6.24

$$\int (\sec^2 \theta) e^{\tan \theta} d\theta, \quad u = \tan \theta$$

$$\frac{du}{d\theta} = \sec^2 \theta \Rightarrow d\theta = \frac{du}{\sec^2 \theta}$$

$$\int (\sec^2 \theta) e^{\tan \theta} d\theta = \int (\sec^2 \theta) e^u \cdot \frac{du}{\sec^2 \theta}$$

$$= \int e^u du$$

$$= e^u + C$$

$$= \boxed{e^{\tan \theta} + C}$$

5.6.26

$$\int \frac{(\ln x)^2 dx}{x}, \quad u = \ln x$$

$$\frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x du$$

$$\int \frac{(\ln x)^2 dx}{x} = \int \frac{u^2}{x} \cdot x du$$

$$= \int u^2 du$$

$$= \frac{u^3}{3} + C$$

$$= \boxed{\frac{(\ln x)^3}{3} + C}$$

5.6.34 In Exercises 34, 40, 47, and 60, evaluate the indefinite integral.

$$\int x^2(x^3+1)^3 dx$$

Let  $u = x^3 + 1$ , so that

$$\frac{du}{dx} = 3x^2 \Rightarrow dx = \frac{du}{3x^2}$$

Therefore

$$\int x^2(x^3+1)^3 dx = \int x^2 \cdot u^3 \cdot \frac{du}{3x^2}$$

$$= \frac{1}{3} \int u^3 du$$

$$= \frac{1}{3} \cdot \frac{u^4}{4} + C$$

$$= \boxed{\frac{(x^3+1)^4}{12} + C}$$

5.6.40

$$\int \frac{x}{\sqrt{x^2+9}} dx$$

$$u = x^2 + 9$$

$$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$$

$$\int \frac{x}{\sqrt{x^2+9}} dx = \int \frac{x}{\sqrt{u}} \cdot \frac{du}{2x}$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

$$= \frac{1}{2} \cdot \frac{u^{1/2}}{1/2} + C$$

$$= \sqrt{u} + C$$

$$= \boxed{\sqrt{x^2+9} + C}$$

5.6.47

$$\int x(x+1)^{1/4} dx$$

$$u = x+1 \Rightarrow x = u-1$$

$$\frac{du}{dx} = 1 \Rightarrow du = dx$$

$$\begin{aligned} \int x(x+1)^{1/4} dx &= \int (u-1)u^{1/4} du \\ &= \int (u^{5/4} - u^{1/4}) du \\ &= \frac{u^{9/4}}{9/4} - \frac{u^{5/4}}{5/4} + C \\ &= \frac{4}{9} u^{9/4} - \frac{4}{5} u^{5/4} + C \\ &= \boxed{\frac{4}{9} (x+1)^{9/4} - \frac{4}{5} (x+1)^{5/4} + C} \end{aligned}$$

5.6.60

$$\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$$

$$u = \sqrt{x}$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow dx = 2\sqrt{x} du$$

$$\begin{aligned} \int \frac{\cos \sqrt{x}}{\sqrt{x}} dx &= \int \frac{\cos u}{\sqrt{x}} \cdot 2\sqrt{x} du \\ &= 2 \int \cos u du \\ &= 2 \sin u + C \\ &= \boxed{2 \sin \sqrt{x} + C} \end{aligned}$$

5.6. 83

In Exercises 83, 84, and 91, use the Change of Variables Formula to evaluate the definite integral.

$$\int_0^2 \frac{x+3}{(x^2+6x+1)^3} dx$$

Let  $u = x^2 + 6x + 1$ , so that

$$\frac{du}{dx} = 2x + 6 \Rightarrow dx = \frac{du}{2x+6} = \frac{du}{2(x+3)}$$

so by the Change of Variables Formula,

$$\int_0^2 \frac{x+3}{(x^2+6x+1)^3} dx = \int_{u(0)}^{u(2)} \frac{x+3}{u^3} \cdot \frac{du}{2(x+3)}$$

$$= \int_{u(0)}^{u(2)} \frac{du}{2u^3}$$

$$= \frac{1}{2} \int_1^{17} \frac{du}{u^3}$$

$$= \frac{1}{2} \cdot \frac{u^{-2}}{-2} \Big|_1^{17}$$

$$= \frac{-1}{4u^2} \Big|_1^{17}$$

$$= -\frac{1}{4} \left( \frac{1}{289} - 1 \right)$$

$$= -\frac{1}{4} \left( \frac{-288}{289} \right)$$

$$= \boxed{\frac{72}{289}}$$

5.6. 84

$$\int_1^2 (x+1)(x^2+2x)^3 dx$$

Let  $u = x^2 + 2x$ , so that

$$\frac{du}{dx} = 2x + 2 \Rightarrow dx = \frac{du}{2x+2} = \frac{du}{2(x+1)}$$

$$= \int_1^2 (x+1)(x^2+2x)^3 dx = \int_{u(1)}^{u(2)} (x+1)u^3 \cdot \frac{du}{2(x+1)}$$

$$= \int_3^8 \frac{u^3}{2} du$$

$$\begin{aligned}
 &= \frac{u^4}{8} \Big|_3^8 \\
 &= 512 - \frac{81}{8} \\
 &= \boxed{\frac{4015}{8}}
 \end{aligned}$$

5.6.91

$$\int_0^{\pi/4} \tan^2 x \sec^2 x \, dx$$

Let  $u = \tan x$ , so that

$$\frac{du}{dx} = \sec^2 x \Rightarrow dx = \frac{du}{\sec^2 x}$$

Therefore we have that

$$\begin{aligned}
 \int_0^{\pi/4} \tan^2 x \sec^2 x \, dx &= \int_{u(0)}^{u(\pi/4)} u^2 \cdot \sec^2 x \cdot \frac{du}{\sec^2 x} \\
 &= \int_0^1 u^2 \, du \\
 &= \frac{u^3}{3} \Big|_0^1 \\
 &= \boxed{\frac{1}{3}}
 \end{aligned}$$

6.1.3 Let  $f(x) = x$  and  $g(x) = 2 - x^2$ .

(a) Find the points of intersection of the two graphs.

$$f(x) = g(x)$$

$$x = 2 - x^2$$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

$$x = -2; x = 1$$

Thus the graphs intersect at  $(-2, -2)$  and  $(1, 1)$ .

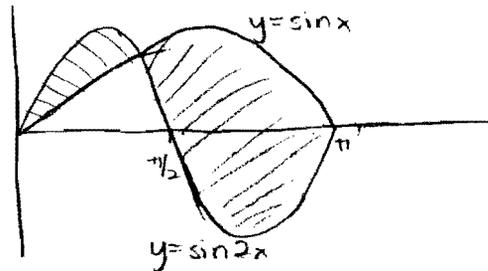
(b) Find the area enclosed by the graphs of  $f$  and  $g$ .

We have that  $g(x) > f(x)$  on  $-2 < x < 1$ , so the area between the curves on this interval is

$$\begin{aligned} A &= \int_{-2}^1 [(2-x^2) - x] dx \\ &= \int_{-2}^1 (2-x-x^2) dx \\ &= \left[ 2x - \frac{x^2}{2} - \frac{x^3}{3} \right]_{-2}^1 \\ &= \left[ \left( 2 - \frac{1}{2} - \frac{1}{3} \right) - \left( -4 - 2 + \frac{8}{3} \right) \right] \\ &= \left[ \frac{7}{6} - \left( -\frac{10}{3} \right) \right] \\ &= \frac{7}{6} + \frac{10}{3} \\ &= \frac{27}{6} = \boxed{\frac{9}{2}} \end{aligned}$$

6.1.16

In Exercises 16 and 18, find the area of the shaded region.



We first find where the graphs intersect in the figure, which is at the first positive value of  $x$  for which  $\sin x = \sin 2x$ :

$$\sin x = \sin 2x = 2 \sin x \cos x$$

$$2 \sin x \cos x - \sin x = 0$$

$$\sin x (2 \cos x - 1) = 0$$

$$\sin x = 0 ; \quad 2 \cos x - 1 = 0 \Rightarrow \cos x = \frac{1}{2}$$

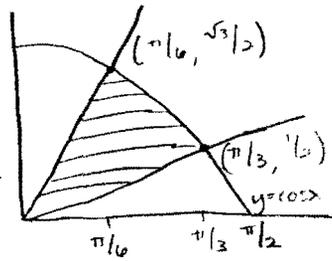
$$\Downarrow \\ x = n\pi, \quad n \in \mathbb{Z};$$

$$\Downarrow \\ x = \frac{\pi}{3}, \quad \frac{5\pi}{6}, \quad \text{etc.}$$

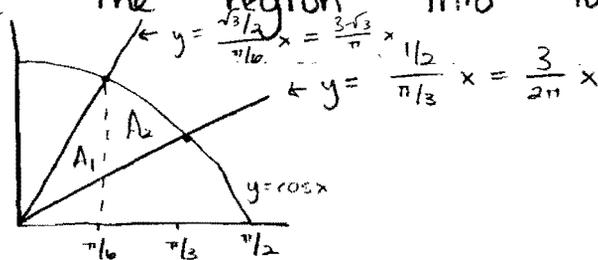
Thus we have that our point of intersection occurs at  $x = \pi/3$ . Because  $\sin 2x > \sin x$  on  $0 < x < \pi/3$  and  $\sin x > \sin 2x$  on  $\pi/3 < x < \pi$ , we have

$$\begin{aligned} \text{Area} &= \int_0^{\pi/3} (\sin 2x - \sin x) dx + \int_{\pi/3}^{\pi} (\sin x - \sin 2x) dx \\ &= \left[ -\frac{1}{2} \cos 2x + \cos x \right]_0^{\pi/3} + \left[ -\cos x + \frac{1}{2} \cos 2x \right]_{\pi/3}^{\pi} \\ &= \left[ \left( -\frac{1}{2} \cos \frac{2\pi}{3} + \cos \frac{\pi}{3} \right) - \left( -\frac{1}{2} \cos 0 + \cos 0 \right) \right] + \\ &\quad \left[ \left( -\cos \pi + \frac{1}{2} \cos 2\pi \right) - \left( -\cos \frac{\pi}{3} + \frac{1}{2} \cos \frac{2\pi}{3} \right) \right] \\ &= \left[ \left( \frac{1}{4} + \frac{1}{2} \right) - \left( -\frac{1}{2} + 1 \right) \right] + \left[ \left( 1 + \frac{1}{2} \right) - \left( -\frac{1}{2} - \frac{1}{4} \right) \right] \\ &= \left[ \frac{3}{4} - \frac{1}{2} \right] + \left[ \frac{3}{2} + \frac{3}{4} \right] \\ &= \frac{1}{4} + \frac{9}{4} \\ &= \boxed{\frac{5}{2}} \end{aligned}$$

6.1.18



We can't do this directly, so we need to divide the region into two parts as follows:



Then the area of the shaded region above will equal  $A_1 + A_2$  But

$$\begin{aligned}
 A_1 &= \int_0^{\pi/6} \left( \frac{3\sqrt{3}}{\pi} x - \frac{3}{2\pi} x \right) dx \\
 &= \int_0^{\pi/6} \left( \frac{3\sqrt{3}}{\pi} - \frac{3}{2\pi} \right) x dx \\
 &= \left( \frac{3\sqrt{3}}{\pi} - \frac{3}{2\pi} \right) \cdot \frac{x^2}{2} \Big|_0^{\pi/6} \\
 &= \left( \frac{6\sqrt{3} - 3}{2\pi} \right) \cdot \left( \frac{\pi^2}{72} - 0 \right) \\
 &= \frac{(6\sqrt{3} - 3)\pi}{48} \\
 &= \frac{(2\sqrt{3} - 1)\pi}{16}
 \end{aligned}$$

$$\begin{aligned}
 A_2 &= \int_{\pi/6}^{\pi/2} \left( \cos x - \frac{3}{2\pi} x \right) dx \\
 &= \left[ \sin x - \frac{3}{4\pi} x^2 \right]_{\pi/6}^{\pi/2} \\
 &= \left( \sin \frac{\pi}{2} - \frac{3}{4\pi} \cdot \frac{\pi^2}{4} \right) - \left( \sin \frac{\pi}{6} - \frac{3}{4\pi} \cdot \frac{\pi^2}{36} \right) \\
 &= \left( \frac{\sqrt{3}}{2} - \frac{\pi}{12} \right) - \left( \frac{1}{2} - \frac{\pi}{48} \right)
 \end{aligned}$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2} - \frac{3\pi}{48}$$

$$= \frac{\sqrt{3}-1}{2} - \frac{\pi}{16}$$

$$\therefore A = A_1 + A_2 = \frac{(2\sqrt{3}-1)\pi}{48} + \frac{\sqrt{3}-1}{2} - \frac{\pi}{16}$$

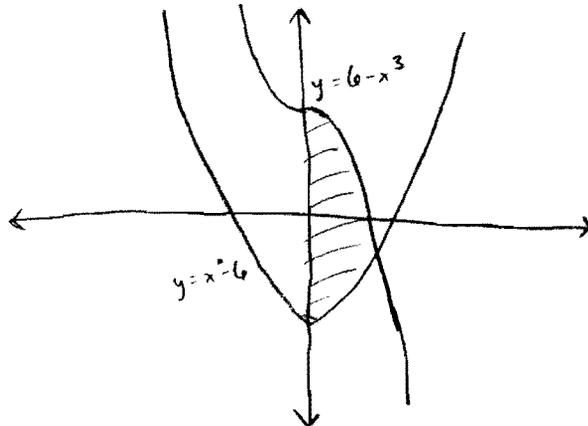
$$= \frac{2\pi\sqrt{3} - \pi + 24\sqrt{3} - 24 - 3\pi}{48}$$

$$= \frac{2\pi\sqrt{3} + 24\sqrt{3} - 24 - 4\pi}{48}$$

$$= \boxed{\frac{12\sqrt{3} - 12 + \pi(\sqrt{3} - 2)}{24}}$$

6.1.30 In Exercises 30, 37, and 42, sketch the region enclosed by the curves and compute its area as an integral along the  $x$ - or  $y$ -axis.

$$y = x^2 - 6, \quad y = 6 - x^3, \quad y\text{-axis}$$



So we need to find the point of intersection of  $x^2 - 6$  and  $6 - x^3$ :

$$x^2 - 6 = 6 - x^3$$

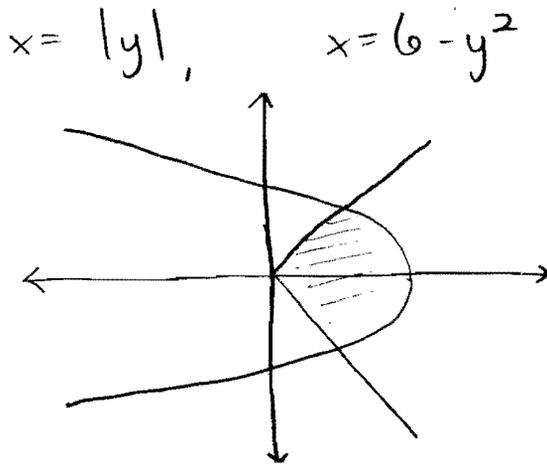
$$x^3 + x^2 - 12 = 0$$

Using a calculator, we find that the only

real solution is  $x = 2$ . Hence the desired area is

$$\begin{aligned}
 A &= \int_0^2 [(6-x^3) - (x^2-6)] dx \\
 &= \int_0^2 (12-x^2-x^3) dx \\
 &= \left[ 12x - \frac{x^3}{3} - \frac{x^4}{4} \right]_0^2 \\
 &= \left( 24 - \frac{8}{3} - 4 \right) - 0 \\
 &= \boxed{\frac{52}{3}}
 \end{aligned}$$

6.1.37



It is best to integrate with respect to  $y$ . First we find points of intersection:

$$y = 6 - y^2 \quad (y \geq 0)$$

$$y^2 + y - 6 = 0$$

$$(y+3)(y-2) = 0$$

$$y = -3; y = 2$$

$$-y = 6 - y^2 \quad (y \leq 0)$$

$$y^2 - y - 6 = 0$$

$$(y-3)(y+2) = 0$$

$$y = 3; y = -2$$

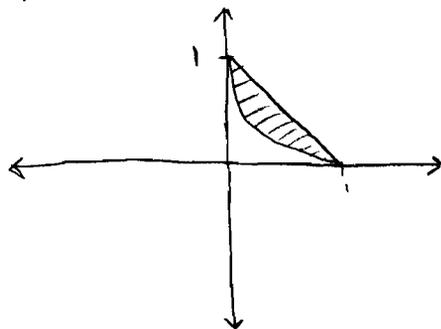
This gives

$$A = \int_{-2}^2 [(6-y^2) - |y|] dy$$

$$\begin{aligned}
 A &= \int_{-2}^0 [(6-y^2) - (-y)] dy + \int_0^2 [(6-y^2) - y] dy \\
 &= \int_{-2}^0 (6-y^2+y) dy + \int_0^2 (6-y^2-y) dy \\
 &= \left[ 6y - \frac{y^3}{3} + \frac{y^2}{2} \right]_{-2}^0 + \left[ 6y - \frac{y^3}{3} - \frac{y^2}{2} \right]_0^2 \\
 &= \left[ 0 - \left( -12 + \frac{8}{3} + 2 \right) \right] + \left[ \left( 12 - \frac{8}{3} - 2 \right) - (0) \right] \\
 &= \frac{22}{3} + \frac{22}{3} \\
 &= \boxed{\frac{44}{3}}
 \end{aligned}$$

6.1.42

$$x+y=1, \quad x^{1/2} + y^{1/2} = 1 \quad \left( y = (1-\sqrt{x})^2 \right)$$



We can find that the two curves intersect at  $(0,1)$  and  $(1,0)$  and that on  $0 < x < 1$ ,  $x+y=1$  lies above  $x^{1/2} + y^{1/2} = 1$ . Thus the area between the two curves is

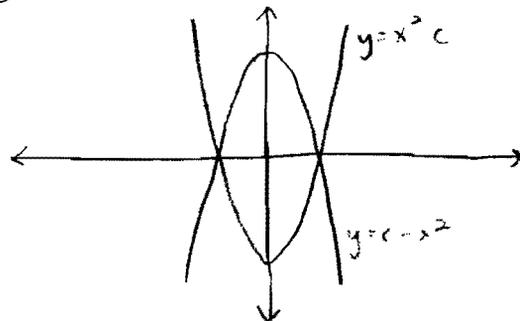
$$\begin{aligned}
 A &= \int_0^1 [(1-x) - (1-\sqrt{x})^2] dx \\
 &= \int_0^1 [1-x - (1-2\sqrt{x}+x)] dx \\
 &= \int_0^1 (-2x + 2\sqrt{x}) dx \\
 &= \left[ -x^2 + \frac{4}{3} x^{3/2} \right]_0^1 \\
 &= \left( -1 + \frac{4}{3} \right) - (0+0) = \boxed{\frac{1}{3}}
 \end{aligned}$$

\* In the next problem, we encounter a common-type exercise, which asks us to find the general solution to an entire family of solutions. If you stick with the mathematics track, you will see that more and more problems of this sort arise, because a common theme of mathematics is to answer questions in as much generality as possible (this in turn often makes a "real-world" problem just a simple application of the general solution).

6.1.50

Find the area enclosed by the curves  $y = c - x^2$  and  $y = x^2 - c$  as a function of  $c$ . Find the value of  $c$  for which this area equals 1.

First we sketch:



Next we find intersection points:

$$x^2 - c = c - x^2$$

$$2x^2 = 2c$$

$$x^2 = c \Rightarrow x = \pm \sqrt{c}$$

Thus

$$\begin{aligned} A(c) &= \int_{-\sqrt{c}}^{\sqrt{c}} [(c - x^2) - (x^2 - c)] dx \\ &= \int_{-\sqrt{c}}^{\sqrt{c}} [2c - 2x^2] dx \end{aligned}$$

$$\begin{aligned} A(c) &= \left[ 2cx - \frac{2x^3}{3} \right]_{-\sqrt{c}}^{\sqrt{c}} \\ &= \left( 2c\sqrt{c} - \frac{2c^{3/2}}{3} \right) - \left( -2c\sqrt{c} + \frac{2c^{3/2}}{3} \right) \\ &= \frac{4}{3}c^{3/2} - \left( -\frac{4}{3}c^{3/2} \right) \end{aligned}$$

$$A(c) = \frac{8}{3}c^{3/2}$$

To find when  $A(c) = 1$ , we have

$$A(c) = 1 = \frac{8}{3}c^{3/2}$$

$$\frac{3}{8} = c^{3/2}$$

$$\left( \frac{3}{8} \right)^{2/3} = c$$