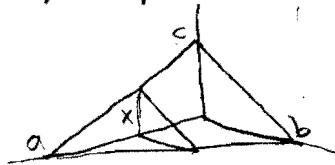


## MATH 20B HOMEWORK 2

- Section 6.2: 7, 9, 12, 13, 52, 53, 57
- Section 6.3: 9, 11, 13, 16, 23, 25, 26, 27, 45, 46, 48, 50

6.2.7 Derive a formula for the volume of the wedge in the figure in terms of the constants  $a$ ,  $b$ , and  $c$ .



We can see that since our cross-sections are taken perpendicular to the  $x$ -axis from  $x=0$  to  $x=a$ , we will have that

$$V = \int_0^a A(x) dx$$

where  $A(x)$  is the area of a cross-section for a given value of  $x$ .

The line from  $c$  to  $a$  is given by  $z = (\frac{c}{a})x + c$ , and the line from  $b$  to  $a$  is given by  $y = (\frac{-b}{a})x + b$ . The cross-sections in question for a given value of  $x$  are (right) triangles with base  $y = (\frac{-b}{a})x + b$ , and height  $z = (\frac{c}{a})x + c$ .

Therefore from the formula for area of a triangle,

$$\begin{aligned} A(x) &= \frac{1}{2} \left[ \left( -\frac{b}{a}x + b \right) \left( \frac{c}{a}x + c \right) \right] \\ &= \frac{bc}{2} \left[ 1 - \frac{x}{a} \right] \left[ 1 + \frac{x}{a} \right] \\ &= \frac{bc}{2} \left[ 1 - \frac{2x}{a} + \frac{x^2}{a^2} \right] \end{aligned}$$

which gives us

$$\begin{aligned} V &= \int_0^a \frac{bc}{2} \left( 1 - \frac{2x}{a} + \frac{x^2}{a^2} \right) dx \\ &= \frac{bc}{2} \left[ x - \frac{x^2}{a} + \frac{x^3}{3a^2} \right]_0^a \\ &= \frac{bc}{2} \left[ \left( a - \frac{a^2}{a} + \frac{a^3}{3a^2} \right) - (0 - 0 + 0) \right] \\ &= \frac{bc}{2} \left[ a - a + \frac{a}{3} \right] \\ &= \frac{bc}{2} \left( \frac{a}{3} \right) \\ &= \boxed{\frac{abc}{6}} \end{aligned}$$

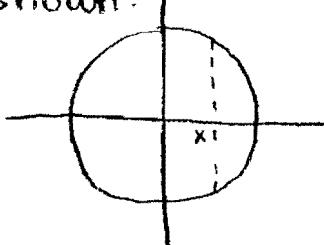
- 6.2.9 In Exercises 9, 12, and 13, find the volume of the solid with given base and cross-sections.

The base is the unit circle  $x^2 + y^2 = 1$  and the cross-sections perpendicular to the  $x$ -axis are triangles whose height and base are equal.

We are taking cross-sections perpendicular to the  $x$ -axis, meaning that our integral will be of the form

$$V = \int_{-1}^1 A(x) dx$$

Let's next think about the base of the triangle. Drawing the unit circle, the base of the cross-section for a given value of  $x$  will be as shown:

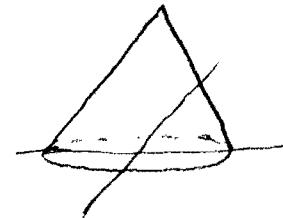


The top part of the circle is given by  $y = \sqrt{1-x^2}$  and the bottom is given by  $y = -\sqrt{1-x^2}$ .

Therefore the base for given  $x$  will have length  $2\sqrt{1-x^2}$ . But we know that the height is equal to the base, so that by the formula for area of a triangle

$$\begin{aligned} A(x) &= \frac{1}{2} \cdot (2\sqrt{1-x^2}) \cdot (2\sqrt{1-x^2}) \\ &= 2(1-x^2) \\ &= 2-2x^2 \end{aligned}$$

$$\begin{aligned} V &= \int_{-1}^1 (2-2x^2) dx \\ &= \left[ 2x - \frac{2x^3}{3} \right]_{-1}^1 \\ &= \left( 2 - \frac{2}{3} \right) - \left( -2 + \frac{2}{3} \right) \\ &= \left( \frac{4}{3} \right) - \left( -\frac{4}{3} \right) \\ &= \boxed{\frac{8}{3}} \end{aligned}$$

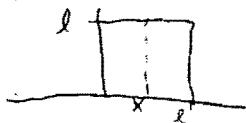


6.2.12 The base is a square, one of whose sides is the interval  $[0, l]$  along the  $x$ -axis. The cross sections perpendicular to the  $x$ -axis are rectangles of height  $f(x) = x^2$ .

Again, we have

$$V = \int_0^l A(x) dx$$

Drawing the base, we see that one side of the rectangle of any cross-section has length  $l$ :



The other side of the cross section's rectangle is equal to  $f(x) = x^2$ . This yields

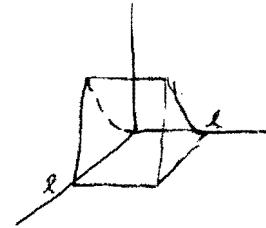
$$A(x) = (l)(x^2)$$

$$\therefore V = \int_0^l lx^2 dx$$

$$= l \cdot \frac{x^3}{3} \Big|_0^l$$

$$= l \cdot \left( \frac{l^3}{3} - 0 \right)$$

$$= \boxed{\frac{l^4}{3}}$$

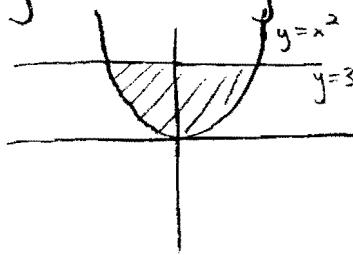


- 6.2.13 The base is the region enclosed by  $y = x^2$  and  $y = 3$ . The cross-sections perpendicular to the  $y$ -axis are squares.

The cross-sections will be perpendicular to the  $y$ -axis, so we will have

$$V = \int_a^b A(y) dy$$

for some values of  $a$  and  $b$ , which we determine by drawing the base:



So we see that  $y$  ranges from 0 to 3. Now, for any value of  $y$ , the length of the segment perpendicular to the  $y$ -axis through the base is equal to

$$\sqrt{y} - (-\sqrt{y}) = 2\sqrt{y}$$

Therefore both sides of the square forming the cross section at  $y$  have length  $2\sqrt{y}$ . By the formula for area of a square, then,

$$A(y) = (2\sqrt{y})^2$$

$$= 4y$$

$$\therefore V = \int_0^3 4y \, dy$$

$$= 2y^2 \Big|_0^3$$

$$= 2(3^2 - 0^2)$$

$$= \boxed{18}$$

- 6.2.52 Find the average value of  $f(x) = ax + b$  over the interval  $[-M, M]$  where  $a, b$  and  $M$  are arbitrary constants.

The formula for average value (provided on p. 387) gives

$$\begin{aligned} \text{Average Value} &= \frac{1}{M - (-M)} \int_{-M}^M f(x) \, dx \\ &= \frac{1}{2M} \int_{-M}^M (ax + b) \, dx \\ &= \frac{1}{2M} \left[ \frac{a}{2}x^2 + bx \right]_{-M}^M \\ &= \frac{1}{2M} \left[ \left( \frac{a}{2}M^2 + bM \right) - \left( \frac{a}{2}(-M)^2 + b(-M) \right) \right] \\ &= \frac{1}{2M} [2bM] \\ &= \boxed{b} \end{aligned}$$

(Note that the constants were labeled in such a way as to purposefully be confusing in order to make sure you really understood what was going on.)

- 6.2.53. The temperature  $T(t)$  at time  $t$  (in hours) in an art museum varies according to

$$T(t) = 70 + 5 \cos\left(\frac{\pi}{12}t\right)$$

Find the average over the time periods  $[0, 24]$  and  $[2, 6]$ .

Again we simply apply the formula for average value:

$$\begin{aligned}\text{Average temp. over } [0, 24] &= \frac{1}{24-0} \int_0^{24} \left[ 70 + 5 \cos\left(\frac{\pi}{12}t\right) \right] dt \\ &= \frac{1}{24} \left[ 70t + \frac{60}{\pi} \sin\left(\frac{\pi}{12}t\right) \right]_0^{24} \\ &= \frac{1}{24} \left[ (70 \cdot 24 + \frac{60}{\pi} \sin(2\pi)) - (0+0) \right] \\ &= \frac{1}{24} [70 \cdot 24 + 0] \\ &= \boxed{70^{\circ}\text{F}}\end{aligned}$$

$$\begin{aligned}\text{Average temp. over } [2, 6] &= \frac{1}{6-2} \int_2^6 \left[ 70 + 5 \cos\left(\frac{\pi}{12}t\right) \right] dt \\ &= \frac{1}{4} \left[ 70t + \frac{60}{\pi} \sin\left(\frac{\pi}{12}t\right) \right]_2^6 \\ &\quad : \\ &= 72.4^{\circ}\text{F}\end{aligned}$$

6.2.57 The acceleration of a particle is

$$a(t) = t - t^3 \frac{m}{s^2} \text{ for } 0 \leq t \leq 1$$

Compute the average acceleration and average velocity over the time interval  $[0, 1]$ , assuming that the particle's initial velocity is zero.

We first need a formula for the particle's velocity, obtained by integrating  $a(t)$ :

$$\begin{aligned} v(t) &= \int a(t) dt \\ &= \int (t - t^3) dt \\ &= \frac{t^2}{2} - \frac{t^4}{4} + C \end{aligned}$$

But  $v(0) = 0$ , which gives  $C = 0$ , so

$$v(t) = \frac{t^2}{2} - \frac{t^4}{4}$$

Now we can find average acceleration and velocity on  $[0, 1]$ :

$$\begin{aligned} \text{Average Acceleration} &= \frac{1}{1-0} \int_0^1 (t - t^3) dt \\ &= \left[ \frac{t^2}{2} - \frac{t^4}{4} \right]_0^1 \\ &= \left( \frac{1}{2} - \frac{1}{4} \right) - (0-0) \\ &= \boxed{\frac{1}{4} \text{ m/s}^2} \end{aligned}$$

$$\begin{aligned} \text{Average Velocity} &= \frac{1}{1-0} \int_0^1 \left( \frac{t^2}{2} - \frac{t^4}{4} \right) dt \\ &= \left[ \frac{t^3}{6} - \frac{t^5}{20} \right]_0^1 \\ &= \left( \frac{1}{6} - \frac{1}{20} \right) - (0-0) = \boxed{\frac{7}{60} \text{ m/s}} \end{aligned}$$

6.3.9 In Exercises 9 and 11, find the volume obtained by rotating the region under the graph of the function about the  $x$ -axis over the given interval.

$$f(x) = \frac{2}{x+1} \quad [1, 3]$$

By the formula for volume of a solid of rotation,

$$\begin{aligned} V &= \pi \int_1^3 f(x)^2 dx \\ &= \pi \int_1^3 \left(\frac{2}{x+1}\right)^2 dx \\ &= 4\pi \int_1^3 (x+1)^{-2} dx \\ &= -4\pi (x+1)^{-1} \Big|_1^3 \\ &= -4\pi \left[ (3+1)^{-1} - (1+1)^{-1} \right] \\ &= -4\pi \left[ \frac{1}{4} - \frac{1}{2} \right] \\ &= -4\pi \left[ -\frac{1}{4} \right] \\ &= \boxed{\pi} \end{aligned}$$

6.3.11  $f(x) = e^x \quad [0, 1]$

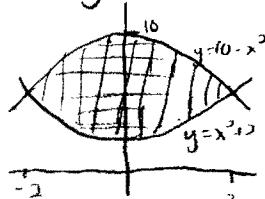
$$\begin{aligned} V &= \pi \int_0^1 (e^x)^2 dx \\ &= \pi \int_0^1 e^{2x} dx \\ &= \frac{\pi}{2} e^{2x} \Big|_0^1 = \frac{\pi}{2} [e^2 - e^0] = \boxed{\frac{\pi}{2} (e^2 - 1)} \end{aligned}$$

6.3.13

In Exercises 13 and 16, (a) sketch the region enclosed by the curves, (b) describe the cross section perpendicular to the  $x$ -axis located at  $x$ , and (c) find the volume of the solid obtained by rotating the region about the  $x$ -axis.

$$y = x^2 + 2, \quad y = 10 - x^2$$

- (a) Setting  $x^2 + 2 = 10 - x^2$  yields  $2x^2 = 8$ , or  $x = \pm 2$ . The region enclosed by the two curves is shown below.



- (b) The cross section perpendicular to the  $x$ -axis located at  $x$  is the segment whose outer radius is  $R(x) = 10 - x^2$  and whose inner radius is  $r(x) = x^2 + 2$ .

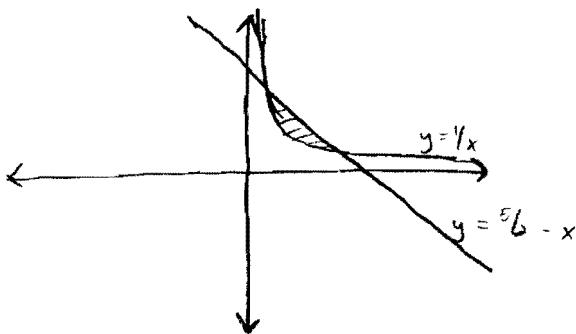
- (c) We use the washer method to find the volume of the solid obtained by rotating the region about the  $x$ -axis:

$$\begin{aligned} V &= \pi \int_{-2}^2 [R(x)^2 - r(x)^2] dx \\ &= \pi \int_{-2}^2 [(10 - x^2)^2 - (x^2 + 2)^2] dx \\ &= \pi \int_{-2}^2 [(100 - 20x^2 + x^4) - (x^4 + 4x^2 + 4)] dx \\ &= \pi \int_{-2}^2 (96 - 24x^2) dx \\ &= \pi [96x - 8x^3]_{-2}^2 = \boxed{256\pi} \end{aligned}$$

6.3.16

$$y = \frac{1}{x}, \quad y = \frac{5}{2} - x$$

(a)



(b) The cross-section perpendicular to the  $x$ -axis located at  $x$  has outer radius  $R(x) = \frac{5}{2} - x$  and inner radius  $r(x) = \frac{1}{x}$ .

(c) We use the washer method to find the volume of the solid. We find the intersection points of our two curves in order to determine the integral of integration:

$$\frac{1}{x} = \frac{5}{2} - x$$

$$1 = \frac{5}{2}x - x^2 \Rightarrow x^2 - \frac{5}{2}x + 1 = 0$$

$$(x-2)(x-\frac{1}{2}) = 0 \Rightarrow x = \frac{1}{2}, x = 2$$

$$V = \pi \int_{1/2}^2 [R(x)^2 - r(x)^2] dx$$

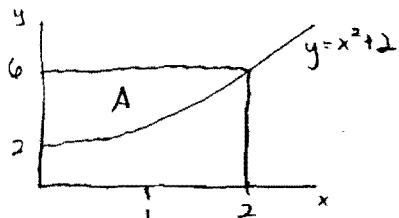
$$= \pi \int_{1/2}^2 \left[ \left( \frac{5}{2} - x \right)^2 - \left( \frac{1}{x} \right)^2 \right] dx$$

$$= \pi \int_{1/2}^2 \left[ \left( \frac{25}{4} - 5x + x^2 \right) - \left( \frac{1}{x^2} \right) \right] dx$$

$$= \pi \left[ \frac{25}{4}x - \frac{5}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{x} \right]_{1/2}^2$$

$$= \boxed{\frac{9\pi}{8}}$$

In Exercises 23, 25, 26, and 27, find the volume of the solid obtained by rotating region A in the figure about the given axis.



6.3.23

x-axis

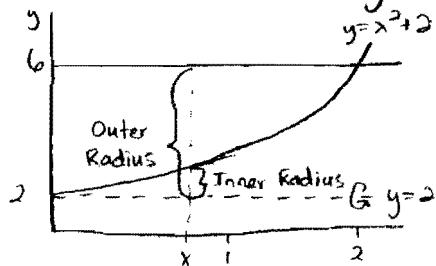
When we rotate about the x-axis, we will use the washer method with inner radius  $r(x) = x^2 + 2$  and  $R(x) = 6$ :

$$\begin{aligned}
 V &= \pi \int_0^2 [R(x)^2 - r(x)^2] dx \\
 &= \pi \int_0^2 [6^2 - (x^2 + 2)^2] dx \\
 &= \pi \int_0^2 [36 - (x^4 + 4x^2 + 4)] dx \\
 &= \pi \int_0^2 (32 - x^4 - 4x^2) dx \\
 &= \pi \left[ 32x - \frac{x^5}{5} - \frac{4}{3}x^3 \right]_0^2 \\
 &= \boxed{\frac{704\pi}{15}}
 \end{aligned}$$

6.3.25

$$y=2$$

When we rotate region A about  $y=2$ , we will apply the washer method, so we need formulas for the inner radius and outer radius for a given  $x$ .



The inner radius,  $r(x)$ , is just the distance from  $y = x^2 + 2$  to the line  $y = 2$ , so

$$r(x) = (x^2 + 2) - 2 = x^2$$

The outer radius,  $R(x)$ , is the distance from  $y = 2$  to  $y = 6$ , so

$$R(x) = 6 - 2 = 4$$

Therefore since our interval is  $0 \leq x \leq 2$ , we apply the Washer Method to obtain

$$V = \pi \int_0^2 [R(x)^2 - r(x)^2] dx$$

$$= \pi \int_0^2 [4^2 - (x^2)^2] dx$$

$$= \pi \int_0^2 (16 - x^4) dx$$

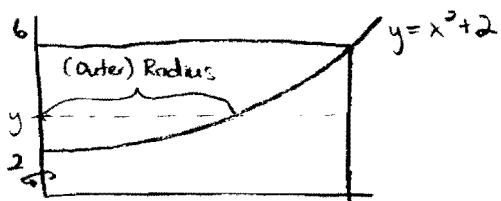
$$= \pi \left[ 16x - \frac{x^5}{5} \right]_0^2$$

$$= \boxed{\frac{128\pi}{5}}$$

6.3.26.

y-axis

Again, we want to use the Washer Method. Here, since we are rotating our inner and outer radius about the y-axis, are with respect to y:



Our inner radius is 0, so in fact we can just use the disk method. Our radius will be the distance from  $x=0$  to the function  $y=x^2+2$ , so solving for  $x$  in terms of  $y$ ,

$$y-2=x^2$$

$$x=\sqrt{y-2}=R(y)$$

Thus since our interval of integration is  $2 \leq y \leq 6$ ,

$$\begin{aligned} V &= \pi \int_2^6 R(y)^2 dy \\ &= \pi \int_2^6 (\sqrt{y-2})^2 dy \\ &= \pi \int_2^6 (y-2) dy \\ &= \pi \left[ \frac{y^2}{2} - 2y \right]_2^6 \\ &= \boxed{8\pi} \end{aligned}$$

\* Note: Why did we ignore  $-\sqrt{y-2}$  when we solved for  $x$  above?

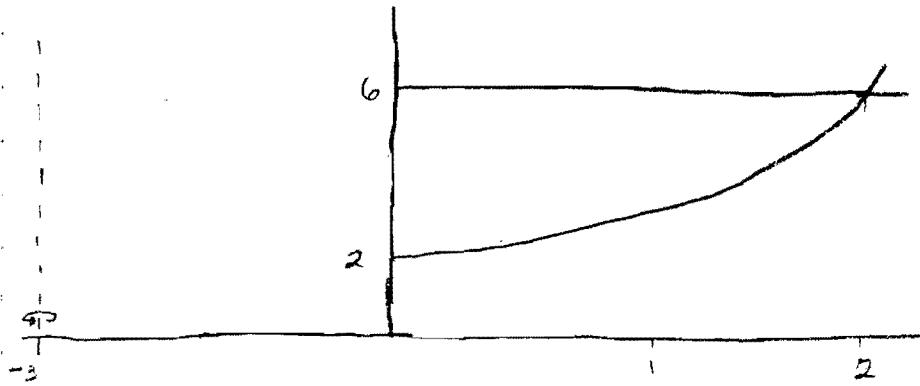
6.3.27

$$x = -3$$

We will have the Washer Method with  
 $r(y) = 3$

$$R(y) = 3 + \sqrt{y-2}$$

$$\begin{aligned} V &= \pi \int_2^6 [R(y)^2 - r(y)^2] dy \\ &= \pi \int_2^6 [(3 + \sqrt{y-2})^2 - 3^2] dy \\ &= \pi \int_2^6 [(9 + 6\sqrt{y-2} + (y-2)) - 9] dy \\ &= \pi \int_2^6 [6\sqrt{y-2} + y - 2] dy \\ &= \pi \left[ 4(y-2)^{3/2} + \frac{1}{2}y^2 - 2y \right]_2^6 \\ &= \boxed{40\pi} \end{aligned}$$

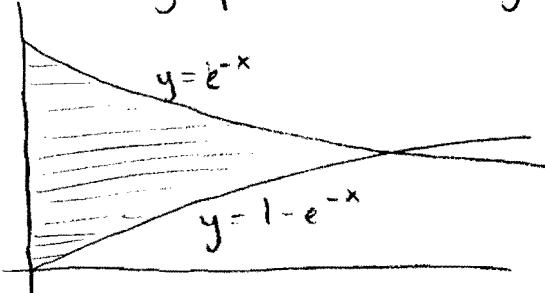


In Exercises 45, 46, and 48, find the volume of the solid obtained by rotating the region enclosed by the graphs about the given axis.

6.3.45

$$y = e^{-x}, \quad y = 1 - e^{-x}, \quad x=0, \quad \text{about } y=4$$

First we graph the region:



So we first need to find the intersection points of  $y = e^{-x}$  and  $y = 1 - e^{-x}$ :

$$e^{-x} = 1 - e^{-x}$$

$$2e^{-x} = 1$$

$$-x = \ln\left(\frac{1}{2}\right) = \ln 1 - \ln 2 = -\ln 2$$

$$\therefore x = \ln 2$$

So our interval of integration is  $[0, \ln 2]$ . As for the inner and outer radii, we have that the outer radius is the distance from  $y=4$  to  $y=1-e^{-x}$ , or  $R(x) = 4 - (1 - e^{-x}) = 3 + e^{-x}$ . The inner radius is the distance from  $y=4$  to  $y=e^{-x}$ , or  $r(x) = 4 - e^{-x}$ .

Hence applying the Washer Method yields

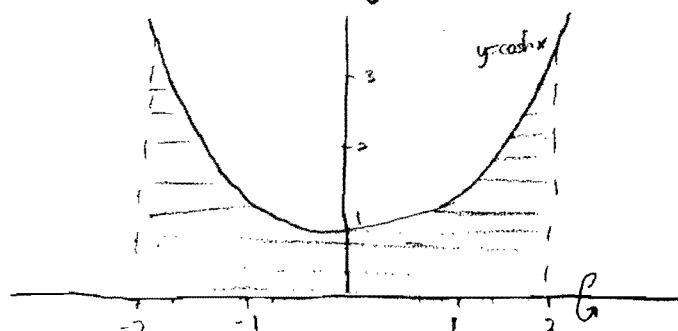
$$V = \pi \int_0^{\ln 2} [R(x)^2 - r(x)^2] dx$$

$$\begin{aligned}
 V &= \pi \int_0^{\ln 2} [(3+e^{-x})^2 - (4-e^{-x})^2] dx \\
 &= \pi \int_0^{\ln 2} [(9+6e^{-x}+e^{-2x}) - (16-8e^{-x}+e^{-2x})] dx \\
 &= \pi \int_0^{\ln 2} (-7+14e^{-x}) dx \\
 &= \pi \left[ -7x - 14e^{-x} \right]_0^{\ln 2} \\
 &= \pi \left[ (-7\ln 2 - 14e^{-\ln 2}) - (0 - 14e^0) \right] \\
 &= \pi \left[ (-7\ln 2 - 14e^{\ln(2)}) - (-14) \right] \\
 &= \pi \left[ -7\ln 2 - 14 \cdot \left(\frac{1}{2}\right) + 14 \right] \\
 &= \pi(-7\ln 2 + 7) \\
 &= \boxed{7\pi(1 - \ln 2)}
 \end{aligned}$$

6.3.46

$$y = \cosh x, \quad x = \pm 2, \quad \text{about } x\text{-axis}$$

We graph the region first:



Thus when we rotate the region about the x-axis we will obtain a solid region whose cross sections perpendicular to the x-axis at x have radius  $R(x) = \cosh x$ . Applying the disk method gives

$$\begin{aligned}
 V &= \pi \int_{-2}^2 R(x)^2 dx \\
 &= \pi \int_{-2}^2 [\cosh(x)]^2 dx \\
 &= \pi \int_{-2}^2 \cosh^2(x) dx
 \end{aligned}$$

We now use the identity

$$\cosh^2(x) = \frac{1}{2}(1 + \cosh 2x)$$

to obtain

$$\begin{aligned}
 V &= \frac{\pi}{2} \int_{-2}^2 (1 + \cosh 2x) dx \\
 &= \frac{\pi}{2} \left[ x + \frac{1}{2} \sinh 2x \right]_{-2}^2 \\
 &= \frac{\pi}{2} \left[ \left( 2 + \frac{1}{2} \sinh 4 \right) - \left( -2 + \frac{1}{2} \sinh(-4) \right) \right] \\
 &= \boxed{\frac{\pi}{2} \left[ 4 + \frac{1}{2} \sinh 4 - \frac{1}{2} \sinh(-4) \right]}
 \end{aligned}$$

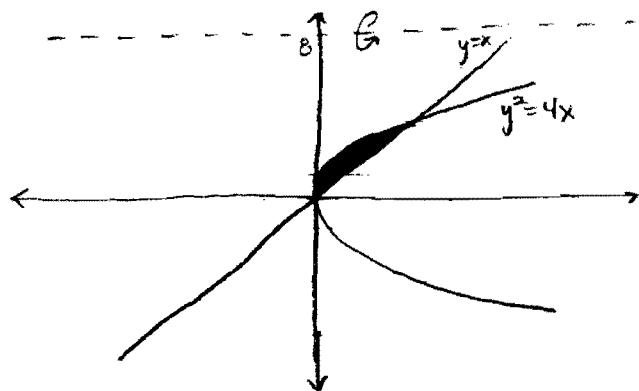
Note: We can simplify further based off the fact that  $\sinh x$  is an odd function to obtain  $\sinh(-4) = -\sinh(4)$ , which gives

$$\begin{aligned}
 V &= \frac{\pi}{2} \left[ 4 + \frac{1}{2} \sinh(4) + \frac{1}{2} \sinh(4) \right] \\
 &= \boxed{\frac{\pi}{2} \left[ 4 + \sinh(4) \right]}
 \end{aligned}$$

6.3.48

$$y^2 = 4x, \quad y = x, \quad \text{about } y = 8$$

We first graph the region.



We therefore need to use the washer method.  
Our inner radius will be the distance from  
 $y = 8$  to  $y^2 = 4x$  for given  $x$ , or

$$r(x) = 8 - \sqrt{4x}$$

Our outer radius will be the distance from  
 $y = 8$  to  $y = x$ , or

$$R(x) = 8 - x$$

To determine the interval of integration, we  
find intersection points of  $y^2 = 4x$  and  $y = x$ :

$$\sqrt{4x} = x$$

$$4x = x^2$$

$$x^2 - 4x = 0$$

$$x(x - 4) = 0$$

$$x = 0, \quad x = 4$$

this gives us  $0 \leq x \leq 4$  as our interval of integration,  
so

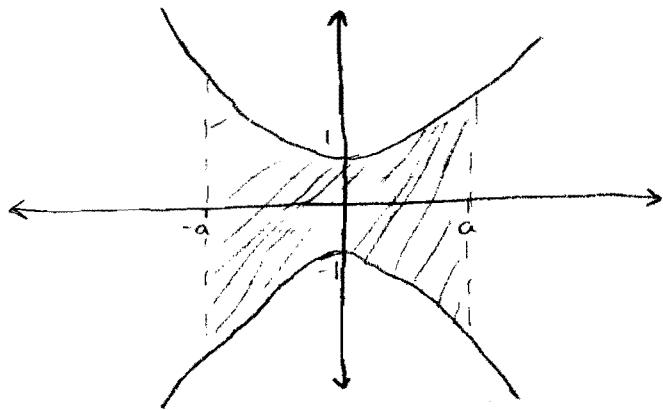
$$\begin{aligned} V &= \pi \int_0^4 [R(x)^2 - r(x)^2] dx \\ &= \pi \int_0^4 [(8-x)^2 - (8-\sqrt{4x})^2] dx \\ &= \pi \int_0^4 [(64 - 16x + x^2) - (64 - 16\sqrt{4x} + 4x)] dx \\ &= \pi \int_0^4 (16\sqrt{4x} - 20x + x^2) dx \end{aligned}$$

Note  $\sqrt{4x} = 2\sqrt{x}$ , so

$$\begin{aligned} V &= \pi \int_0^4 (32\sqrt{x} - 20x + x^2) dx \\ &= \pi \left[ 32 \cdot \frac{x^{3/2}}{3/2} - 10x^2 + \frac{x^3}{3} \right]_0^4 \\ &= \pi \left[ \frac{64}{3}x^{3/2} - 10x^2 + \frac{x^3}{3} \right]_0^4 \\ &= \pi \left[ \left( \frac{64}{3} \cdot 8 - 160 + \frac{64}{3} \right) - (0) \right] \\ &= \boxed{32\pi} \end{aligned}$$

6.3.50 The solid obtained by rotating the region between the branches of the hyperbola  $y^2 - x^2 = 1$  about the  $x$ -axis is called a hyperboloid. Find the volume of the hyperboloid for  $-a \leq x \leq a$ .

Graphing  $y^2 - x^2 = 1$  gives



Rotating this region about the  $x$ -axis gives a solid region whose cross sections perpendicular to the  $x$ -axis have radius

$$R(x) = \sqrt{1+x^2} \quad \left( \text{since } y^2 - x^2 = 1 \Rightarrow y^2 = 1+x^2 \right)$$

Therefore the volume of the hyperboloid for  $-a \leq x \leq a$  is obtained by the disk method:

$$\begin{aligned} V &= \pi \int_{-a}^a R(x)^2 dx \\ &= \pi \int_{-a}^a (\sqrt{1+x^2})^2 dx \\ &= \pi \int_{-a}^a (1+x^2) dx \\ &= \pi \left[ x + \frac{x^3}{3} \right]_{-a}^a \\ &= \pi \left[ \left( a + \frac{a^3}{3} \right) - \left( -a - \frac{a^3}{3} \right) \right] = \boxed{\pi \left( \frac{2a^3}{3} + 2a \right)} \end{aligned}$$