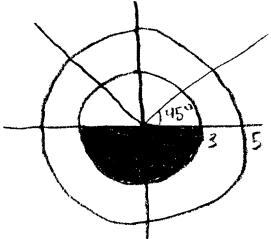


MATH 20B HOMEWORK 3 SOLUTIONS

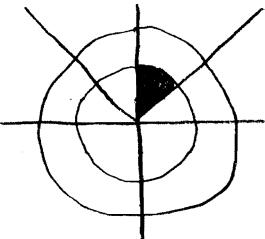
- Section 11.3: 8, 10, 15, 17, 18, 21, 22, 24, 25, 35, 36, 37, 38
- Section 11.4: 5, 6, 7, 8, 12, 13, 16, 18
 - Note: Skip Arc Length (p. 652)
- Supplement 1.6: 1, 2, 3, 4

11.3.8

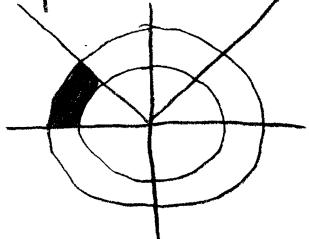
Describe each shaded sector by inequalities in r and θ .



(A)



(B)



(C)

$$(A): 0 \leq r \leq 3, \quad \pi \leq \theta \leq 2\pi$$

$$(B): 0 < r \leq 3, \quad \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$$

$$(C): 3 \leq r \leq 5, \quad \frac{3\pi}{4} \leq \theta \leq \pi$$

11.3.10

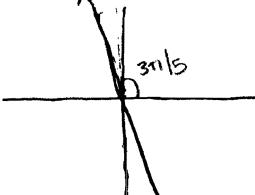
What is the slope of the line $\theta = \frac{3\pi}{5}$?

We know that $\theta = \tan^{-1}(y/x)$, so

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{3\pi}{5}$$

$$\frac{y}{x} = \tan\left(\frac{3\pi}{5}\right)$$

$$y = \tan\left(\frac{3\pi}{5}\right)x$$



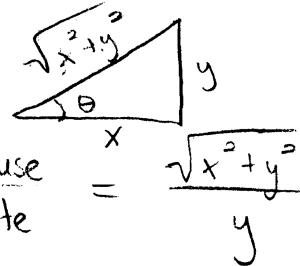
$$m = \tan\left(\frac{3\pi}{5}\right) \approx -3.1$$

In Exercises 15 and 17, convert to an equation in rectangular coordinates.

11.3.15

$$r = 2 \csc \theta$$

Drawing a triangle can help,
since $\theta = \tan^{-1}(y/x)$:



$$\therefore \csc \theta = \frac{1}{\sin \theta} = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{\sqrt{x^2 + y^2}}{y}$$

Recall $r = \sqrt{x^2 + y^2}$, so
 $\sqrt{x^2 + y^2} = 2 \cdot \frac{\sqrt{x^2 + y^2}}{y}$

$$\begin{aligned} 1 &= \frac{2}{y} \\ \therefore y &= 2 \end{aligned}$$

11.3.17

$$r = \frac{1}{2 - \cos \theta}$$

We could draw a triangle for this one, but for an alternate method recall $x = r \cos \theta$ so $\cos \theta = x/r = x/\sqrt{x^2 + y^2}$, giving

$$\sqrt{x^2 + y^2} = \frac{1}{2 - x/\sqrt{x^2 + y^2}}$$

$$\sqrt{x^2 + y^2} = \frac{1}{(2\sqrt{x^2 + y^2} - x)/\sqrt{x^2 + y^2}}$$

$$\sqrt{x^2 + y^2} = \frac{\sqrt{x^2 + y^2}}{2\sqrt{x^2 + y^2} - x}$$

$$2\sqrt{x^2 + y^2} - x = 1$$

$$(2\sqrt{x^2 + y^2})^2 = (x+1)^2$$

$$4(x^2 + y^2) = x^2 + 2x + 1$$

$$3x^2 - 2x + 4y^2 = 1$$

$$3\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + 4y^2 = 1 + \frac{1}{3}$$

$$3\left(x - \frac{1}{3}\right)^2 + 4y^2 = \frac{4}{3}$$

$$\boxed{\frac{(x - 1/3)^2}{4/9} + \frac{y^2}{4/3} = 1}$$

(Ellipse centered at $(1/3, 0)$)

In Exercises 18 and 21, convert to an equation in polar coordinates.

11.3.18

$$x^2 + y^2 = 5$$

$$\therefore r^2 = 5$$

$$\boxed{\therefore r = \sqrt{5}}$$

11.3.21

$$xy = 1$$

$$(r\cos\theta)(r\sin\theta) = 1$$

$$r^2 \sin\theta \cos\theta = 1$$

$$r^2 \sin 2\theta = 1$$

$$\boxed{r^2 = 2 \csc 2\theta}$$

11.3.22

Match the equation with its description:

(a) $r=2$ (i) Vertical Line

(b) $\theta=2$ (ii) Horizontal Line

(c) $r = 2 \sec \theta$ (iii) Circle

(d) $r = 2 \csc \theta$ (iv) Line Through Origin

In (a),

$$r=2 \Rightarrow \sqrt{x^2+y^2} = 2 \Rightarrow x^2+y^2=4 \quad (\text{Circle})$$

$$(b) \theta = 2 \Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = 2$$

$$\frac{y}{x} = \tan(2)$$

$y = \tan(2)x$ (Line through origin)

$$(c) r = 2 \sec \theta$$

$$r = \frac{2}{\cos \theta}$$

$$r \cos \theta = 2$$

$$x = 2 \quad (\text{Vertical line})$$

$$(d) r = 2 \csc \theta$$

$$r = \frac{2}{\sin \theta}$$

$$r \sin \theta = 2$$

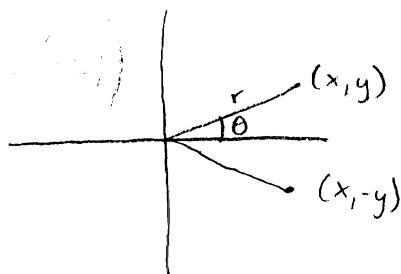
$$y = 2 \quad (\text{Horizontal line})$$

II. 3.24

Suppose that (x, y) has polar coordinates (r, θ) .
Find the polar coordinates of the following:

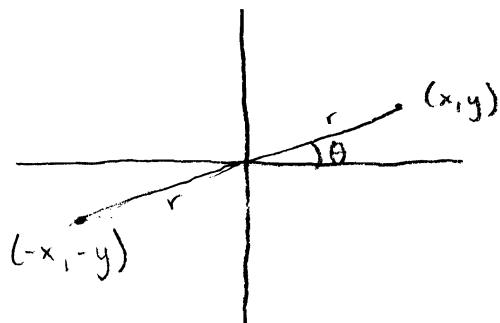
(a) $(x, -y) = (r, -\theta)$

(or $(r, 2n - \theta)$)



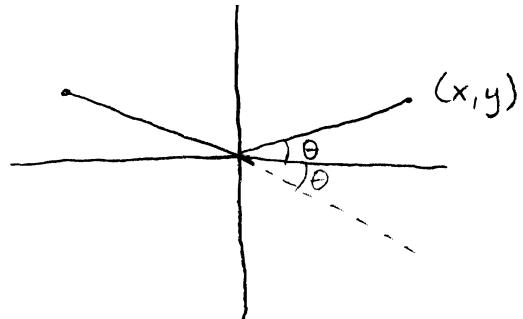
(b) $(-x, -y) = (-r, \theta)$

(or $(r, \theta + \pi)$)

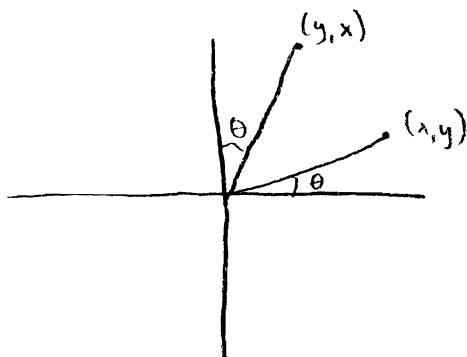


(c) $(-x, y) = (-r, -\theta)$

(or $r, \pi - \theta$)



(d) $(y, x) = \left(r, \frac{\pi}{2} - \theta\right)$



11.3.25

Match each equation in rectangular coordinates with its equation in polar coordinates:

$$(a) \quad x^2 + y^2 = 2$$

$$\therefore r^2 = 2 \Rightarrow r = \sqrt{2} \quad (\text{matches with (iv), error in book})$$

$$(b) \quad x^2 + (y - 1)^2 = 1$$

$$x^2 + y^2 - 2y + 1 = 1$$

$$x^2 + y^2 - 2y = 0$$

$$r^2 - 2rsin\theta = 0 \Rightarrow r^2 = 2rsin\theta \Rightarrow r = 2sin\theta \quad (\text{iii})$$

$$(c) \quad x^2 - y^2 = 4$$

$$(rcos\theta)^2 - (rsin\theta)^2 = 4$$

$$r^2(\cos^2\theta - \sin^2\theta) = 4$$

$$r^2((1 - \sin^2\theta) - \sin^2\theta) = 4 \Rightarrow r^2(1 - 2\sin^2\theta) = 4 \quad (\text{ii})$$

(ERROR IN BOOK)

$$(d) \quad x + y = 4$$

$$rcos\theta + rsin\theta = 4$$

$$r(\cos\theta + \sin\theta) = 4 \quad (\text{ii})$$

11.3.35

Show that $r = a\cos\theta + b\sin\theta$ is the equation of a circle passing through the origin.

We first show that this gives a circle:

$$r = a\cos\theta + b\sin\theta$$

$$r^2 = r(a\cos\theta + b\sin\theta)$$

$$x^2 + y^2 = a \cdot r\cos\theta + b \cdot r\sin\theta$$

$$x^2 + y^2 = ax + by$$

$$(x^2 - ax) + (y^2 - by) = 0$$

$$\left(x^2 - ax + \frac{a^2}{4}\right) + \left(y^2 - by + \frac{b^2}{4}\right) = \frac{a^2}{4} + \frac{b^2}{4}$$

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \frac{a^2 + b^2}{4}$$

This is the equation of a circle with center $(\frac{a}{2}, \frac{b}{2})$ and radius $\sqrt{\frac{a^2 + b^2}{4}} = \frac{\sqrt{a^2 + b^2}}{2}$.

Next, we show that it passes through the origin; it suffices to show that $(0,0)$ is a solution of our equation:

$$\left(0 - \frac{a}{2}\right)^2 + \left(0 - \frac{b}{2}\right)^2 = \frac{a^2}{4} + \frac{b^2}{4} = \frac{a^2 + b^2}{4} \quad \checkmark$$

11.3.36

Use the previous exercise to write the equation of the circle of radius 5 and center $(3, 4)$ in the form $r = a\cos\theta + b\sin\theta$.

From 11.3.35, we must have that the center is at $(\frac{a}{2}, \frac{b}{2})$, so

$$\frac{a}{2} = 3 \quad , \quad \frac{b}{2} = 4$$

$$\therefore a = 6 \quad \therefore b = 8$$

So we have

$$r = 6\cos\theta + 8\sin\theta$$

11.3.37

Use the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$ to find a polar equation of the hyperbola $x^2 - y^2 = 1$.

$$x^2 - y^2 = 1$$

$$r^2 \cos^2\theta - r^2 \sin^2\theta = 1$$

$$r^2 (\cos^2\theta - \sin^2\theta) = 1$$

$$\therefore r^2 (\cos 2\theta) = 1 \quad \text{or}$$

$$r^2 = \sec 2\theta$$

11.3.38

Find an equation in rectangular coordinates for the curves $r^2 = \cos 2\theta$:

We use the identity $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$:

$$r^2 = \cos^2 \theta - \sin^2 \theta$$

Multiply both sides by r^2 :

$$r^4 = r^2 \cos^2 \theta - r^2 \sin^2 \theta$$

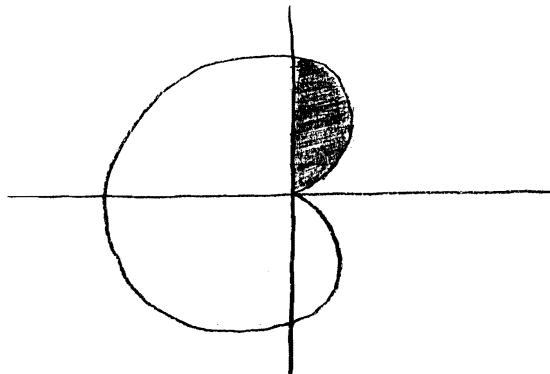
$$r^4 = (r \cos \theta)^2 - (r \sin \theta)^2$$

$$(r^2)^2 = (r \cos \theta)^2 - (r \sin \theta)^2$$

$$\boxed{(x^2 + y^2)^2 = x^2 - y^2}$$

11.4.6

Find the total area enclosed by the cardioid $r = 1 - \cos \theta$:



The shaded area is the region swept out by the cardioid as θ ranges from 0 to 2π (note that $r > 0$ on this interval), so

$$A = \frac{1}{2} \int_0^{2\pi} (1 - \cos \theta)^2 d\theta$$

$$A = \frac{1}{2} \int_0^{2\pi} (1 - 2\cos \theta + \cos^2 \theta) d\theta$$

$$A = \frac{1}{2} \int_0^{2\pi} d\theta - \int_0^{2\pi} \cos \theta d\theta + \frac{1}{2} \int_0^{2\pi} \cos^2 \theta d\theta$$

$$A = \pi - \left[\sin \theta \right]_0^{2\pi} + \frac{1}{2} \int_0^{2\pi} \frac{1+\cos 2\theta}{2} d\theta$$

$$A = \pi - 0 + \frac{1}{4} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{2\pi}$$

$$A = \pi + \frac{1}{4} \left[\left(2\pi + \frac{0}{2} \right) - \left(0 + \frac{0}{2} \right) \right]$$

$$A = \pi + \frac{\pi}{2}$$

$$\boxed{A = \frac{3\pi}{2}}$$

II.4.6 Find the area of the shaded region in the figure in II.4.5.

The integral is the same, we just integrate on $0 \leq \theta \leq \frac{\pi}{2}$ (note $r > 0$ on this interval):

$$A = \int_0^{\frac{\pi}{2}} (1-\cos \theta)^2 d\theta$$

$$A = \frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta - \int_0^{\frac{\pi}{2}} \cos \theta d\theta + \frac{1}{4} \int_0^{\frac{\pi}{2}} (1+\cos 2\theta) d\theta$$

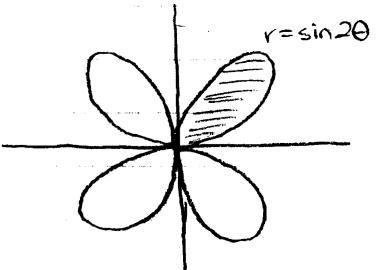
$$A = \left[\frac{\theta}{2} - \sin \theta + \frac{\theta}{4} + \frac{\sin 2\theta}{8} \right]_0^{\frac{\pi}{2}}$$

$$A = \left(\frac{\pi}{4} - 1 + \frac{\pi}{8} + 0 \right) - (0 - 0 + 0 + 0)$$

$$\boxed{A = \frac{3\pi}{8} - 1}$$

II. 4.7

Find the area of one leaf of the "four-petaled rose" $r = \sin 2\theta$:



On the shaded leaf, θ ranges from 0 to $\pi/2$ (and $r > 0$ on this interval), so

$$A = \frac{1}{2} \int_0^{\pi/2} r^2 d\theta$$

$$A = \frac{1}{2} \int_0^{\pi/2} (\sin 2\theta)^2 d\theta$$

$$A = \frac{1}{2} \int_0^{\pi/2} \left(\frac{1 - \cos 4\theta}{2} \right) d\theta$$

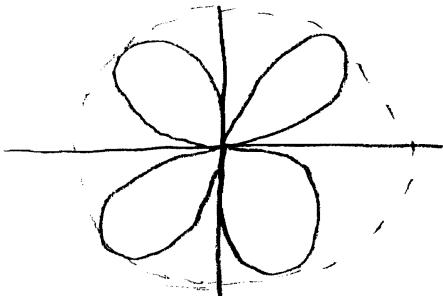
$$A = \frac{1}{4} \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/2}$$

$$A = \frac{1}{4} \left[\left(\frac{\pi}{2} - 0 \right) + (0 - 0) \right]$$

$$\boxed{A = \frac{\pi}{8}}$$

II. 4.8

Prove that the total area of the four-petaled rose $r = \sin 2\theta$ is equal to one-half the area of the circumscribed circle:



On one hand, the area of the rose is

$$A_{\text{rose}} = \int_0^{2\pi} (\sin 2\theta)^2 d\theta = \frac{\pi}{2}$$

(which we can obtain by either going through the integration again or by multiplying 4 times the area of a petal found in Exercise 7).

On the other hand to find the area of the circumscribed circle we must find when r is maximal, so we use calculus:

$$r = \sin 2\theta$$

$$\therefore \frac{dr}{d\theta} = 2\cos 2\theta = 0$$

$$\cos 2\theta = 0$$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ etc.}$$

$$\frac{d^2r}{d\theta^2} = -4\sin 2\theta$$

$$\left. \frac{d^2r}{d\theta^2} \right|_{\theta=\pi/4} = -4 \sin\left(\frac{\pi}{2}\right) = -4$$

Therefore r has a local maximum at $\theta = \pi/4$, where $r = \sin(2 \cdot \pi/4) = \sin(\pi/2) = 1$. So our circle has radius 1 and

$$A(\text{circle}) = \pi(1)^2 = \pi$$

Observe that this is twice the area of the rose, which is what we set out to show.

11.4.12

Find the area of the intersection of the circles $r = \sin \theta$ and $r = \cos \theta$.

It helps to change to rectangular coordinates in order to graph the circles:

$$r = \sin \theta$$

$$r = \cos \theta$$

$$r^2 = r \sin \theta$$

$$r^2 = r \cos \theta$$

$$x^2 + y^2 = y$$

$$x^2 + y^2 = x$$

$$x^2 + (y^2 - y) = 0$$

$$(x^2 - x) + y^2 = 0$$

$$x^2 + (y^2 - y + \frac{1}{4}) = \frac{1}{4}$$

$$(x^2 - x + \frac{1}{4}) + y^2 = \frac{1}{4}$$

$$x^2 + (y - \frac{1}{2})^2 = (\frac{1}{2})^2$$

$$(x - \frac{1}{2})^2 + y^2 = (\frac{1}{2})^2$$

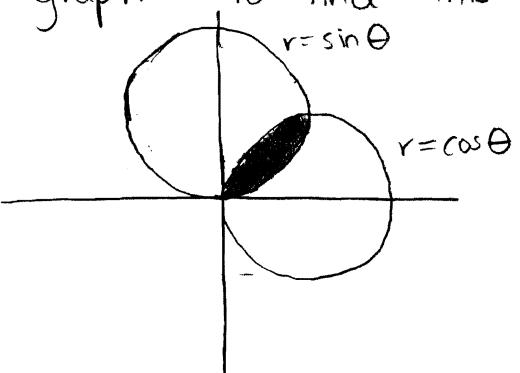
Center: $(0, \frac{1}{2})$

Center: $(\frac{1}{2}, 0)$

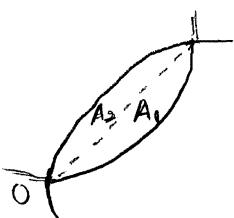
Radius: $\frac{1}{2}$

Radius: $\frac{1}{2}$

Now we can graph to find this area of intersection:



Let's zoom in on this region:



We need to divide the region down the middle in order to split the integral into two parts that we can work with.

We next need to find the points of intersection:

$$r = \sin \theta = \cos \theta$$

$$\theta = \frac{\pi}{4}$$

This accounts for the intersection point farther from the origin. The curves also intersect at the origin; $\sin \theta = 0$ when $\theta = 0$ and $\cos \theta = 0$ when $\theta = \frac{\pi}{2}$.

So the idea is to integrate with respect to $\sin \theta$ for $0 < \theta \leq \frac{\pi}{4}$ and $\cos \theta$ for $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$. (Confusion at this point is justifiable. Thinking of 'each' curve completely separately can help). So

$$A_1 = \frac{1}{2} \int_0^{\pi/4} (\sin \theta)^2 d\theta$$

$$= \frac{1}{2} \int_0^{\pi/4} \left(\frac{1 - \cos 2\theta}{2} \right) d\theta$$

$$= \frac{1}{4} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/4}$$

$$= \frac{1}{4} \left[\left(\frac{\pi}{4} - \frac{1}{2} \right) - (0 - 0) \right]$$

$$= \frac{\pi}{16} - \frac{1}{8}$$

$$A_2 = \frac{1}{2} \int_{\pi/4}^{\pi/2} (\cos \theta)^2 d\theta$$

$$= \frac{1}{2} \int_{\pi/4}^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta$$

$$= \frac{1}{4} \left[\theta + \frac{\sin 2\theta}{2} \right]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{4} \left[\left(\frac{\pi}{2} + 0 \right) - \left(\frac{\pi}{4} + \frac{1}{2} \right) \right]$$

$$= \frac{\pi}{16} - \frac{1}{8}$$

Adding the two yields the area of the region:

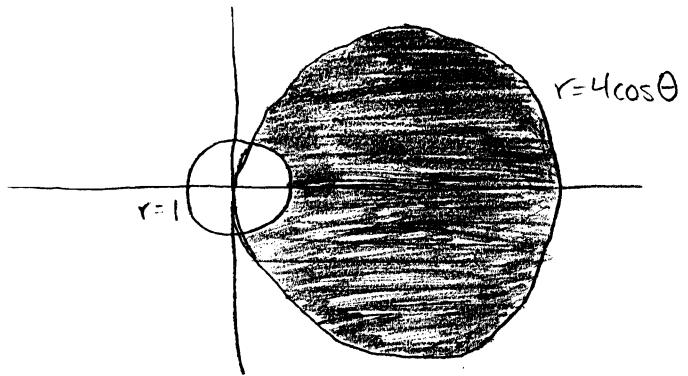
$$A = A_1 + A_2$$

$$A = \left(\frac{\pi}{16} - \frac{1}{8} \right) + \left(\frac{\pi}{16} - \frac{1}{8} \right)$$

$$A = \frac{\pi}{8} - \frac{1}{4}$$

II. 4.13

Find the area of the shaded region:



Here we can actually use the formula for area between two curves, with $r_1 = 4\cos\theta$ and $r_2 = 1$. We just need the values θ_1 and θ_2 at which the curves intersect:

$$4\cos\theta = 1$$

$$\cos\theta = \frac{1}{4}$$

$$\theta_1 = -\cos^{-1}\left(\frac{1}{4}\right), \quad \theta_2 = \cos^{-1}\left(\frac{1}{4}\right)$$

Therefore

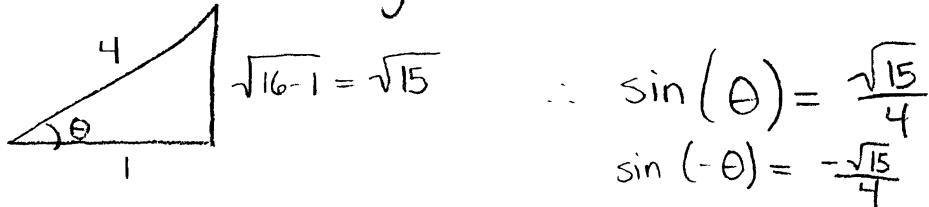
$$\begin{aligned} A &= \frac{1}{2} \int_{-\cos^{-1}(1/4)}^{\cos^{-1}(1/4)} \left[(4\cos\theta)^2 - 1^2 \right] d\theta \\ &= \frac{1}{2} \int_{-\cos^{-1}(1/4)}^{\cos^{-1}(1/4)} (16\cos^2\theta - 1) d\theta \\ &= \frac{1}{2} \int_{-\cos^{-1}(1/4)}^{\cos^{-1}(1/4)} \left(\frac{16(\cos 2\theta + 1)}{2} - 1 \right) d\theta \\ &= \int_{-\cos^{-1}(1/4)}^{\cos^{-1}(1/4)} \left(4\cos 2\theta + \frac{7}{2} \right) d\theta \\ &= \left[2\sin 2\theta + \frac{7}{2}\theta \right]_{-\cos^{-1}(1/4)}^{\cos^{-1}(1/4)} \\ &= \left[4\sin\theta \cos\theta + \frac{7}{2}\theta \right]_{-\cos^{-1}(1/4)}^{\cos^{-1}(1/4)} \end{aligned}$$

$$= \left[4 \sin(\cos^{-1}(\frac{1}{4})) \cos(\cos^{-1}(\frac{1}{4})) + \frac{7}{2} \cos^{-1}(\frac{1}{4}) \right] - \\ \left[4 \sin(-\cos^{-1}(\frac{1}{4})) \cos(-\cos^{-1}(\frac{1}{4})) + \frac{7}{2} \cdot (-\cos^{-1}(\frac{1}{4})) \right]$$

which reduces to

$$A = 4 \sin(\cos^{-1}(\frac{1}{4})) \cdot \frac{1}{4} - 4 \sin(-\cos^{-1}(\frac{1}{4})) \cdot \frac{1}{4} + 7 \cos^{-1}(\frac{1}{4}) \\ = \sin(\cos^{-1}(\frac{1}{4})) - \sin(-\cos^{-1}(\frac{1}{4})) + 7 \cos^{-1}(\frac{1}{4})$$

We draw a triangle for $\theta = \cos^{-1}(\frac{1}{4})$



so

$$A = \left[\frac{\sqrt{15}}{4} - \left(-\frac{\sqrt{15}}{4} \right) \right] + 7 \cos^{-1}\left(\frac{1}{4}\right)$$

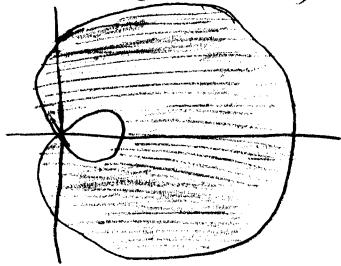
$$A = \frac{\sqrt{15}}{2} + 7 \cos^{-1}\left(\frac{1}{4}\right) \approx 11.163$$

Note 1: If the manipulations involving trig and inverse trig functions seemed complicated, it is because I omitted the details of why some of the steps I used were kosher.

Note 2: This whole thing worked because $r = 4\cos\theta$ traces out the entire circumference of the larger circle in π radians whereas the smaller circle is traced out in 2π radians. In general we would often obtain interval of integration issues, because the shaded region above involves most of the larger circle but not even half of the smaller circle, all because the parametrization by θ just happens to align. The upshot is that we should always be wary of the parametrization.

11.4.16

Find the area between the inner and outer loop of the limacon $r = 2\cos\theta - 1$.



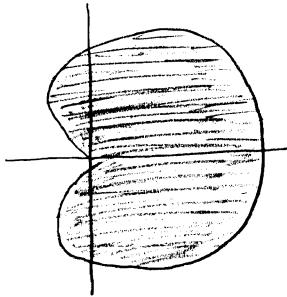
$r = 2\cos\theta - 1$ will equal zero twice as the curve is traced out once:

$$2\cos\theta - 1 = 0$$

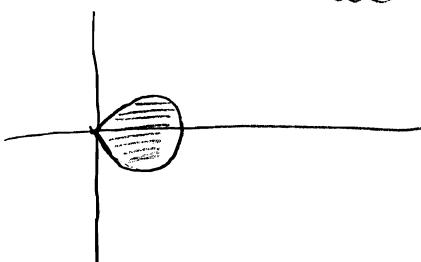
$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}; \theta = \frac{5\pi}{3}$$

Now, when $\theta = 0$, $r = 2\cos\theta - 1$ is 1, whereas when $\theta = \pi$, $r = 2\cos\theta - 1$ is -3, so the outer loop is (perhaps surprisingly) traced out for $\frac{\pi}{3} \leq \theta \leq \frac{5\pi}{3}$. If we take the area A_1 under $r = 2\cos\theta - 1$ for these values we get the region



whereas if we take the area A_2 under $r = 2\cos\theta - 1$ for $-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}$ we get the region



But this is great, because then our desired area A is just $A_1 - A_2$. So

$$\begin{aligned}
 A_1 &= \int_{\pi/3}^{5\pi/3} \frac{1}{2} (2\cos\theta - 1)^2 d\theta \\
 &= \int_{\pi/3}^{5\pi/3} \frac{1}{2} (4\cos^2\theta - 4\cos\theta + 1) d\theta \\
 &= 2 \int_{\pi/3}^{5\pi/3} \cos^2\theta d\theta - 2 \int_{\pi/3}^{5\pi/3} \cos\theta d\theta + \frac{1}{2} \int_{\pi/3}^{5\pi/3} d\theta \\
 &= 2 \int_{\pi/3}^{5\pi/3} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta - 2 \int_{\pi/3}^{5\pi/3} \cos\theta d\theta + \frac{1}{2} \int_{\pi/3}^{5\pi/3} d\theta \\
 &= \left[\theta + \frac{\sin 2\theta}{2} \right]_{\pi/3}^{5\pi/3} - \left[2\sin\theta \right]_{\pi/3}^{5\pi/3} + \left[\frac{\theta}{2} \right]_{\pi/3}^{5\pi/3} \\
 &= \left[\left(\frac{5\pi}{3} + \frac{(-\sqrt{3})}{4} \right) - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{4} \right) \right] - 2 \left[-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \right] + \frac{1}{2} \left[\frac{5\pi}{3} - \frac{\pi}{3} \right] \\
 &= \left[\frac{4\pi}{3} - \frac{\sqrt{3}}{2} \right] + 2\sqrt{3} + \frac{2\pi}{3} \\
 &= 2\pi + \sqrt{3} \left(\frac{-1}{2} + 2 \right) \\
 &= \boxed{2\pi + \frac{3\sqrt{3}}{2}}
 \end{aligned}$$

$$A_2 = \int_{-\pi/3}^{\pi/3} \frac{1}{2} (2\cos\theta - 1)^2 d\theta$$

\therefore (Same procedure)

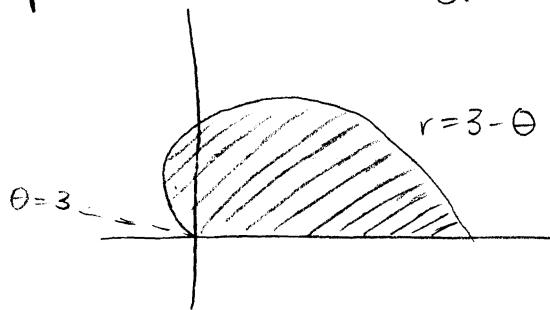
$$\begin{aligned}
 &= \left[\theta + \frac{\sin 2\theta}{2} \right]_{-\pi/3}^{\pi/3} - 2 \left[\sin\theta \right]_{-\pi/3}^{\pi/3} + \frac{1}{2} \left[\theta \right]_{-\pi/3}^{\pi/3} \\
 &= \left[\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4} \right) - \left(-\frac{\pi}{3} - \frac{\sqrt{3}}{4} \right) \right] - 2 \left[\frac{\sqrt{3}}{2} - \frac{(-\sqrt{3})}{2} \right] + \frac{1}{2} \left[\frac{\pi}{3} - \left(-\frac{\pi}{3} \right) \right] \\
 &= \frac{2\pi}{3} + \frac{\sqrt{3}}{2} - 2\sqrt{3} + \frac{\pi}{3} \\
 &= \boxed{\pi + \frac{3\sqrt{3}}{2}}
 \end{aligned}$$

$$\therefore A = \left(2\pi + \frac{3\sqrt{3}}{2} \right) - \left(\pi - \frac{3\sqrt{3}}{2} \right)$$

$$\boxed{A = \pi + 3\sqrt{3}}$$

11.4.18

Compute the area of the shaded region below.



It is very tempting to think that we just integrate on the interval $0 \leq \theta \leq \pi$. Instead, the figure implies that we integrate from $\theta = 0$ to when $r = 0$; that is, $\theta = 3$. So

$$\begin{aligned}
 A &= \frac{1}{2} \int_0^3 (3-\theta)^2 d\theta \\
 &= \frac{1}{2} \int_0^3 (9-6\theta+\theta^2) d\theta \\
 &= \frac{1}{2} \left[9\theta - 3\theta^2 + \frac{\theta^3}{3} \right]_0^3 \\
 &= \frac{1}{2} \left[(27 - 27 + 9) - (0 - 0 + 0) \right] \\
 &= \boxed{\frac{9}{2}}
 \end{aligned}$$

Supp. 1.6.1 Simplify the following and write in rectangular form:

$$(a) \quad 5 - \frac{3}{2}i - \left(8 + \frac{5}{2}i\right) = 5 - 8 + i\left(-\frac{3}{2} - \frac{5}{2}\right)$$
$$= \boxed{-3 + 4i}$$

$$(b) \quad (-2 + 7i)(5 - 4i) = -10 + 8i + 14i - 28i^2$$
$$= (-10 + 28) + i(8 + 14)$$
$$= \boxed{18 + 22i}$$

$$(c) \quad \overline{3i(6-i)} = (\overline{3i})(\overline{6-i})$$
$$= (-3i)(6+i)$$
$$= -18i - 3i^2$$
$$= \boxed{3 - 18i}$$

$$(d) \quad \frac{1+i}{2-3i} = \frac{1+i}{2-3i} \cdot \frac{2+3i}{2+3i}$$
$$= \frac{2+3i+2i+3i^2}{(2-3i)(2+3i)}$$
$$= \frac{2+5i-3}{4+9}$$
$$= \frac{-1+5i}{13}$$
$$= \boxed{\frac{-1}{13} + \frac{5}{13}i}$$

Supp. 1.6.2 Let $z = 1 + \sqrt{3}i$, $w = -1 - i$.

(a) Write z and w in polar form:

$$|z| = (1^2 + (\sqrt{3})^2)^{1/2} \quad |w| = ((-1)^2 + (-1)^2)^{1/2}$$

$$= \sqrt{4}$$

$$= \sqrt{2}$$

$$= 2$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$

$$= \frac{\pi}{3}$$

$$\theta = \tan^{-1}(1) + \pi$$

$$= \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$

$$z = 2 \left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right)$$

$$w = \sqrt{2} \left(\cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right) \right)$$

(b) Compute zw , \bar{z}/w , and $1/z$ (in polar form)

$$zw = 2\sqrt{2} \left[\cos\left(\frac{\pi}{3} + \frac{5\pi}{4}\right) + i \sin\left(\frac{\pi}{3} + \frac{5\pi}{4}\right) \right]$$

$$= 2\sqrt{2} \left[\cos\left(\frac{19\pi}{12}\right) + i \sin\left(\frac{19\pi}{12}\right) \right]$$

$$\frac{z}{w} = \frac{2}{\sqrt{2}} \left[\cos\left(\frac{\pi}{3} - \frac{5\pi}{4}\right) + i \sin\left(\frac{\pi}{3} - \frac{5\pi}{4}\right) \right]$$

$$= \sqrt{2} \left[\cos\left(\frac{-11\pi}{12}\right) + i \sin\left(\frac{-11\pi}{12}\right) \right]$$

$$\frac{1}{z} = \frac{1}{2} \left[\cos\left(\frac{-\pi}{3}\right) + i \sin\left(\frac{-\pi}{3}\right) \right]$$

Supp. 1.6.3 Simplify the following expressions and write the results in the polar form $r(\cos(\theta) + i\sin(\theta))$

$$(a) (1+i)^{13}$$

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\begin{aligned}\therefore (1+i)^{13} &= \sqrt{2}^{13} \left[\cos\left(13 \cdot \frac{\pi}{4}\right) + i\sin\left(13 \cdot \frac{\pi}{4}\right) \right] \\ &= \boxed{2^{13/2} \left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) \right]}\end{aligned}$$

$$(b) (-\sqrt{3}+i)^{15}$$

$$r = \sqrt{(-\sqrt{3})^2 + 1^2} = 2, \quad \theta = \tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right) + \pi = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}$$

$$\begin{aligned}\therefore (-\sqrt{3}+i)^{15} &= 2^{15} \left[\cos\left(15 \cdot \frac{5\pi}{6}\right) + i\sin\left(15 \cdot \frac{5\pi}{6}\right) \right] \\ &= \boxed{2^{15} \left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) \right]}\end{aligned}$$

$$(c) (32-32i)^{-6}$$

$$r = \sqrt{32^2 + (-32)^2} = 32\sqrt{2}, \quad \theta = \tan^{-1}(-1) = -\frac{\pi}{4}$$

$$\begin{aligned}\therefore (32-32i)^{-6} &= (32\sqrt{2})^{-6} \left[\cos\left(-6 \cdot -\frac{\pi}{4}\right) + i\sin\left(-6 \cdot -\frac{\pi}{4}\right) \right] \\ &= \boxed{2^{-33} \left[\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) \right]}\end{aligned}$$

Supp. 1.6.4 Find each of the following roots and sketch them in the complex plane

(a) The cube roots of 1

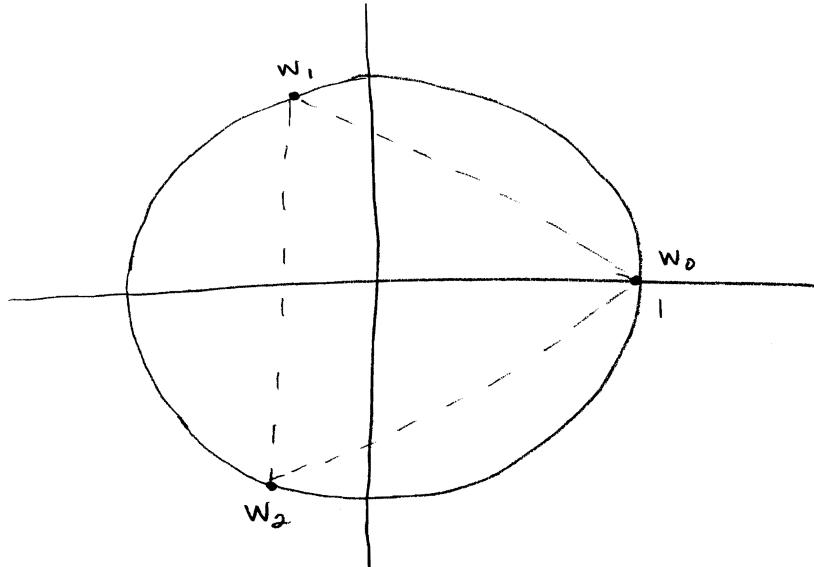
So $w^3 = 1$, which since $1 = 1 \cdot [\cos(0) + i\sin(0)]$ gives

$$w_k = 1 \left[\cos\left(\frac{0+2\pi k}{3}\right) + i\sin\left(\frac{0+2\pi k}{3}\right) \right], \quad k=0,1,2$$

$$\therefore w_0 = 1^{1/3} \left[\cos(0) + i\sin(0) \right] = 1$$

$$w_1 = 1^{1/3} \left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) \right] = -\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}$$

$$w_2 = 1^{1/3} \left[\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) \right] = -\frac{1}{2} - i \cdot \frac{\sqrt{3}}{2}$$



(b) The fourth roots of $1+i$

$$1+i : r = \sqrt{2}, \theta = \frac{\pi}{4}$$

So for $w^4 = 1+i$ we have

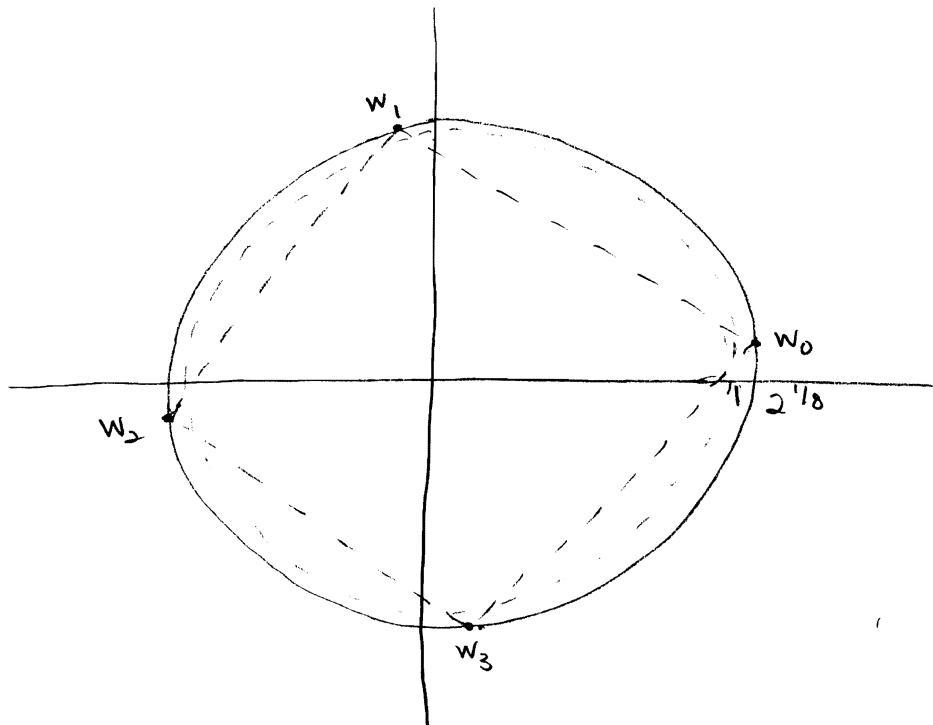
$$w_n = (\sqrt{2})^{1/4} \left[\cos\left(\frac{\pi/4 + 2\pi k}{4}\right) + i \sin\left(\frac{\pi/4 + 2\pi k}{4}\right) \right], \quad n=0,1,2,3$$

$$\therefore w_0 = 2^{1/8} \left[\cos\left(\frac{\pi}{16}\right) + i \sin\left(\frac{\pi}{16}\right) \right]$$

$$w_1 = 2^{1/8} \left[\cos\left(\frac{9\pi}{16}\right) + i \sin\left(\frac{9\pi}{16}\right) \right]$$

$$w_2 = 2^{1/8} \left[\cos\left(\frac{17\pi}{16}\right) + i \sin\left(\frac{17\pi}{16}\right) \right]$$

$$w_3 = 2^{1/8} \left[\cos\left(\frac{25\pi}{16}\right) + i \sin\left(\frac{25\pi}{16}\right) \right]$$



(c) The sixth roots of -64

$$-64 = 64[\cos(\pi) + i\sin(\pi)] \quad (r=64, \theta=\pi)$$

So for $w^6 = -64$, we have

$$w_k = 64^{\frac{1}{6}} \left[\cos\left(\frac{\pi + 2\pi k}{6}\right) + i\sin\left(\frac{\pi + 2\pi k}{6}\right) \right], \quad k=0,1,2,3,4,5$$

$$\therefore w_0 = 2 \left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) \right]$$

$$w_1 = 2 \left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) \right]$$

$$w_2 = 2 \left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) \right]$$

$$w_3 = 2 \left[\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right) \right]$$

$$w_4 = 2 \left[\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) \right]$$

$$w_5 = 2 \left[\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right) \right]$$

