

MATH 20B HW4 SOLUTIONS

- Supplement 2.2: 1a, 2, 3, 4, 5
- Section 7.1: 4, 9, 28a, 31. (skip Error Bound), 33 (skip Error Bound),
- Section 7.2: 13, 21, 36, 39, 42, 44, 46, 47, 54, 63, 66

Supp. 2.2.1a Simplify and write in the polar form $r e^{i\theta}$
 $(\sqrt{3} + i)^5$

If $z = \sqrt{3} + i$, then $r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$. Also,
 $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

so that

$$\begin{aligned}
 \sqrt{3} + i &= 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right) \\
 \therefore (\sqrt{3} + i)^5 &= 2^5 \left(\cos\left(5 \cdot \frac{\pi}{6}\right) + i \cdot \sin\left(5 \cdot \frac{\pi}{6}\right)\right) \\
 &= 32 \left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) \\
 &= \boxed{32 e^{i \cdot \frac{5\pi}{6}}}
 \end{aligned}$$

Supp. 2.2.2

We know that $\cos^3(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$. Use the relationship between the sine, cosine, and exponential functions to express $\cos^3(x)$ as a sum of sines and cosines.

Recall that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

so that cubing both sides we obtain

$$\begin{aligned}\cos^3 x &= \left(\frac{e^{ix} + e^{-ix}}{2}\right)^3 \\ &= \frac{1}{8} \left((e^{ix})^3 + 3(e^{ix})^2(e^{-ix}) + 3(e^{ix})(e^{-ix})^2 + (e^{-ix})^3 \right) \\ &= \frac{1}{8} \left(e^{3ix} + 3(e^{2ix})(e^{-ix}) + 3(e^{ix})(e^{-2ix}) + e^{-3ix} \right) \\ &= \frac{1}{8} \left(e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-3ix} \right)\end{aligned}$$

To get this back to a form that can be written as a sum of trig functions, we regroup terms:

$$\begin{aligned}\cos^3 x &= \frac{1}{8} \left((e^{3ix} + e^{-3ix}) + 3(e^{ix} + e^{-ix}) \right) \\ &= \frac{1}{8} \left(2 \cdot \left(\frac{e^{i(3x)} + e^{-i(3x)}}{2} \right) + 3 \cdot 2 \cdot \left(\frac{e^{ix} + e^{-ix}}{2} \right) \right) \\ &= \frac{1}{8} \left(2\cos(3x) + 6\cos x \right) \\ &= \boxed{\frac{\cos(3x)}{4} + \frac{3\cos x}{4}}\end{aligned}$$

Supp. 2.2.3

Show that $e^{\pi i} + 1 = 0$.

We know that for x real
 $e^{ix} = \cos x + i \sin x$

so that

$$e^{i\pi} = \cos \pi + i \sin \pi$$

but

$$\cos \pi = -1, \sin \pi = 0, \text{ so}$$
$$e^{i\pi} = -1$$

$$\therefore e^{i\pi} + 1 = 0 \quad \checkmark$$

Supp. 2.2.4

What are the Cartesian coordinates $x+iy = e^{2+3i}$ of the complex number $x+iy = e^{2+3i}$?

$$\begin{aligned} e^{2+3i} &= (e^2)(e^{3i}) \\ &= e^2 (\cos(3) + i \sin(3)) \end{aligned}$$

$$= e^2 \cos(3) + i \cdot e^2 \sin(3)$$

Thus $x = e^2 \cos(3), y = e^2 \sin(3)$.

Supp. 2.2.5 Use the fact that

$$\frac{d}{dx} [\cos(bx) + i\sin(bx)] = b[-\sin(bx) + i\cos(bx)]$$

and the product rule to show that

$$\frac{d}{dx} [e^{(a+bi)x}] = (a+bi)e^{(a+bi)x}$$

In order to use what we have been given, we need to get $e^{(a+bi)x}$ in a suitable form.

$$e^{(a+bi)x} = e^{ax} \cdot e^{ibx}$$

$$= e^{ax} \cdot [\cos(bx) + i\sin(bx)]$$

This is good because we are given the derivative of the second term, and the derivative of the first is just ae^{ax} , so we apply the product rule:

$$\begin{aligned}\frac{d}{dx} [e^{(a+bi)x}] &= ae^{ax} [\cos(bx) + i\sin(bx)] + be^{ax} [-\sin(bx) + i\cos(bx)] \\ &= ae^{ax} e^{ibx} + be^{ax} [-\sin(bx) + i\cos(bx)] \\ &= ae^{(a+bi)x} + be^{ax} [-\sin(bx) + i\cos(bx)]\end{aligned}$$

So we need the second term to equal $bie^{(a+bi)x}$.

Let's try multiplying by i/i :

$$\begin{aligned}\frac{d}{dx} [e^{(a+bi)x}] &= ae^{(a+bi)x} + \frac{bie^{ax}}{i} [-\sin(bx) + i\cos(bx)] \\ &= ae^{(a+bi)x} + bie^{ax} \left[-\frac{\sin(bx)}{i} + \cos(bx) \right] \\ &= ae^{(a+bi)x} + bie^{ax} [i\sin(bx) + \cos(bx)] \\ &= ae^{(a+bi)x} + bie^{ax} e^{bix} \\ &= ae^{(a+bi)x} + bie^{(a+bi)x} = (a+bi)e^{(a+bi)x}\end{aligned}$$

In Exercises 7.1.4 and 7.1.9, calculate T_N and M_N for the value of N indicated.

7.1.4

$$\int_0^{\pi/2} \sin \sqrt{x} \, dx \quad N=8 \quad f(x) = \sqrt{x}$$

$$T_N = \frac{\pi/2 - 0}{2 \cdot 8} \left[f(0) + 2f\left(\frac{\pi}{16}\right) + 2f\left(\frac{\pi}{8}\right) + 2f\left(\frac{3\pi}{16}\right) + 2f\left(\frac{\pi}{4}\right) + 2f\left(\frac{5\pi}{16}\right) + 2f\left(\frac{3\pi}{8}\right) + 2f\left(\frac{7\pi}{16}\right) + f\left(\frac{\pi}{2}\right) \right]$$

$$\approx 1.18005$$

$$M_N = \frac{\pi/2 - 0}{8} \left[f\left(\frac{\pi}{32}\right) + f\left(\frac{3\pi}{32}\right) + f\left(\frac{5\pi}{32}\right) + f\left(\frac{7\pi}{32}\right) + f\left(\frac{9\pi}{32}\right) + f\left(\frac{11\pi}{32}\right) + f\left(\frac{13\pi}{32}\right) + f\left(\frac{15\pi}{32}\right) \right]$$

$$\approx 1.20344$$

7.1.9

$$\int_2^3 \frac{dx}{\ln x} \quad N=5 \quad f(x) = \frac{1}{\ln x}$$

$$T_N = \frac{3-2}{2-5} \left[f(2) + 2f(2.2) + 2f(2.4) + 2f(2.6) + 2f(2.8) + f(3) \right]$$

$$\approx 1.12096$$

$$M_N = \frac{3-2}{5} \left[f(2.1) + f(2.3) + f(2.5) + f(2.7) + f(2.9) \right]$$

$$\approx 1.11716$$

7.1.28 (a) Calculate T_6 for the integral

$$I = \int_0^2 x^3 dx$$

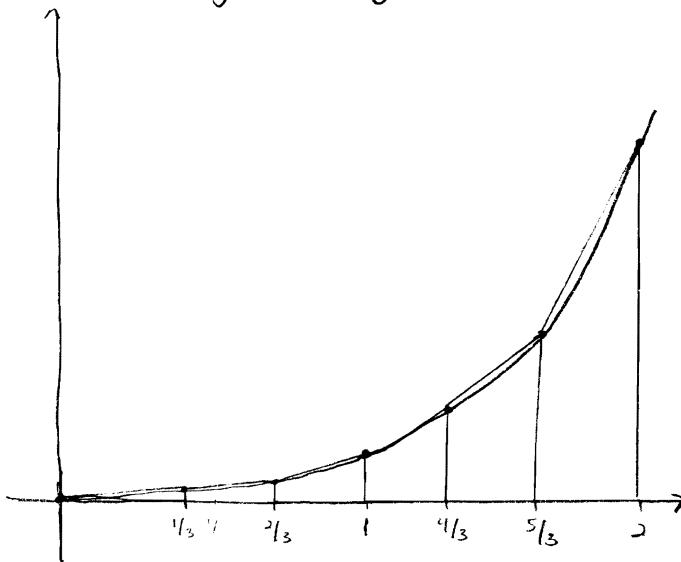
Is T_6 too large or too small? Explain graphically.

$$f(x) = x^3$$

$$T_6 = \frac{2-0}{2-6} \left[f(0) + 2f\left(\frac{1}{3}\right) + 2f\left(\frac{2}{3}\right) + 2f(1) + 2f\left(\frac{4}{3}\right) + 2f\left(\frac{5}{3}\right) + f(2) \right]$$

$$\approx 4.1111$$

T_6 is too large. Strictly speaking, this is because $f(x) = x^3$ is concave up on $[0, 2]$. However, graphically we can see it as follows:



7.1.31

In Exercises 31 and 33, state whether T_N or M_N underestimates or overestimates the integral (but do not calculate T_N or M_N).
 $\int_0^2 e^{-x/4} dx$, T_{20}

We simply determine concavity. If $f(x) = e^{-x/4}$, then

$$f'(x) = -\frac{1}{4} e^{-x/4}$$

$$f''(x) = \frac{1}{16} e^{-x/4}$$

But the exponential function is always positive, so $f''(x) > 0$ for all x . Since f is concave up on all of \mathbb{R} , T_{20} overestimates the integral.

$$\int_0^{\pi/4} \cos x, M_{20}$$

We use the same strategy employed in 31. If $f(x) = \cos x$, then

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

On $[0, \pi/4]$, $\cos x$ is positive; therefore, $f''(x) < 0$ and so f is concave down. This means that M_{20} overestimate the integral.

In Exercises 7.2.13 and 7.2.21, use Integration by Parts to evaluate the integral.

7.2.13

$$\int e^{-x} \sin x \, dx$$

$$u = \sin x$$

$$v' = -e^{-x}$$

$$u' = \cos x$$

$$v = -e^{-x}$$

$$\therefore \int e^{-x} \sin x \, dx = -e^{-x} \sin x + \int e^{-x} \cos x \, dx$$

$$u = \cos x$$

$$v' = e^{-x}$$

$$u' = -\sin x$$

$$v = -e^{-x}$$

$$\therefore \int e^{-x} \sin x \, dx = -e^{-x} \sin x + \left(-e^{-x} \cos x - \int e^{-x} \sin x \, dx \right)$$

$$= -e^{-x} \cos x - e^{-x} \sin x - \int e^{-x} \sin x \, dx$$

$$2 \int e^{-x} \sin x \, dx = -e^{-x} \cos x - e^{-x} \sin x + C$$

$$\boxed{\int e^{-x} \sin x \, dx = \frac{1}{2} e^{-x} (\cos x + \sin x) + C}$$

C

7.2.21

$$\int x \cdot 2^x dx$$

$$u = x, \quad v' = 2^x$$

$$u' = 1, \quad v = \frac{1}{\ln 2} 2^x$$

$$\begin{aligned}\int x \cdot 2^x dx &= \frac{x}{\ln 2} 2^x - \int \frac{2^x}{\ln 2} dx \\ &= \frac{x}{\ln 2} 2^x - \frac{1}{(\ln 2)^2} \cdot 2^x + C \\ &= \boxed{\frac{2^x}{\ln 2} \left(x - \frac{1}{\ln 2} \right) + C}\end{aligned}$$

C

In Exercises 7.2.36, 7.2.39, 7.2.42, and 7.2.44, evaluate using Integration by Parts, substitution, or both if necessary.

7.2.36

$$\int \frac{\ln(\ln x)}{x} dx$$

$$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x du$$

$$\begin{aligned}\int \frac{\ln(\ln x)}{x} dx &= \int \frac{\ln(u)}{x} \cdot x du \\ &= \int \ln u du\end{aligned}$$

$$= u \ln u - u + C \quad (\text{See Example 3, p. 428})$$

$$= \boxed{(\ln x)(\ln(\ln x)) - \ln x + C}$$

C

7.2.39

$$\int \cos x \ln(\sin x) dx$$

$$u = \sin x \Rightarrow \frac{du}{dx} = \cos x \Rightarrow dx = \frac{du}{\cos x}$$

$$\int \cos x \ln(\sin x) dx = \int \cos x \ln u \cdot \frac{du}{\cos x}$$

$$= \int \ln u du$$

$$= u \ln u - u + C$$

$$= \boxed{\sin x \ln(\sin x) - \sin x + C}$$

7.2.42

$$\int \sqrt{x} e^{\sqrt{x}} dx$$

$$w = \sqrt{x} \Rightarrow \frac{dw}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow dx = 2\sqrt{x} dw$$

$$\therefore \int \sqrt{x} e^{\sqrt{x}} dx = \int 2w^2 e^w dw$$

$$u = 2w^2 \Rightarrow u' = 4w, \quad v = e^w \Rightarrow v = e^w$$

$$\therefore \int \sqrt{x} e^{\sqrt{x}} dx = 2w^2 e^w - \int 4we^w dw$$

$$u = 4w \Rightarrow u' = 4 \quad v = e^w \Rightarrow v = e^w$$

$$\therefore \int \sqrt{x} e^{\sqrt{x}} dx = 2w^2 e^w - \left(4we^w - \int 4e^w dw \right)$$

$$= 2w^2 e^w - 4we^w + 4e^w + C$$

$$= \boxed{2x e^{\sqrt{x}} - 4\sqrt{x} e^{\sqrt{x}} + 4e^{\sqrt{x}} + C}$$

7.2.44

$$\int x \tan^{-1} x \, dx$$

$$u = \tan^{-1} x \quad v = x$$

$$u' = \frac{1}{x^2 + 1} \quad v = \frac{x^2}{2}$$

$$\begin{aligned}\therefore \int x \tan^{-1} x \, dx &= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{x^2 + 1} \, dx \\ &= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{x^2 + 1}\right) dx \\ &= \boxed{\frac{x^2}{2} \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + C}\end{aligned}$$

In Exercises 7.2.46 and 7.2.47, compute the definite integral.

7.2.46 $\int_1^3 \ln x \, dx$

$$u = \ln x \quad v' = 1$$

$$u' = \frac{1}{x} \quad v = x$$

$$\begin{aligned}\int_1^3 \ln x \, dx &= x \ln x \Big|_1^3 - \int_1^3 dx \\ &= \left[x \ln x - x \right]_1^3 \\ &= (3 \ln 3 - 3) - (0 - 1) \\ &= \boxed{3 \ln 3 - 2}\end{aligned}$$

7.2.47 $\int_0^4 x \sqrt{4-x} \, dx$

$$u = 4 - x \Rightarrow \frac{du}{dx} = -1 \Rightarrow dx = -du$$

$$\begin{aligned}\int_0^4 x \sqrt{4-x} \, dx &= \int_{x=0}^{x=4} (4-u) \sqrt{u} (-1) du \\ &= \int_4^0 (u^{3/2} - 4u^{1/2}) du \\ &= \int_0^4 (4u^{1/2} - u^{3/2}) du \\ &= \left[\frac{8u^{3/2}}{3} - \frac{2u^{5/2}}{5} \right]_0^4 \\ &= \left(\frac{8 \cdot 8}{3} - \frac{2 \cdot 32}{5} \right) - (0 - 0) \\ &= \frac{64}{3} - \frac{64}{5} = \boxed{\frac{128}{15}}\end{aligned}$$

7.2.54

Evaluate $\int x^n \ln x \, dx$ for $n \neq -1$. Which method should be used to evaluate $\int x^{-1} \ln x \, dx$?

We can use u-substitution to evaluate $\int x^{-1} \ln x \, dx$. As for the case $n \neq -1$, we have

$$u = \ln x \quad v = x^n$$

$$u' = \frac{1}{x} \quad v = \frac{x^{n+1}}{n+1}$$

$$\begin{aligned}\therefore \int x^n \ln x \, dx &= \ln x \cdot \frac{x^{n+1}}{n+1} - \int \frac{x^{n+1}}{n+1} \cdot \frac{1}{x} \, dx \\&= \ln x \cdot \frac{x^{n+1}}{n+1} - \int \frac{x^n}{n+1} \, dx \\&= \ln x \cdot \frac{x^{n+1}}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C \\&= \boxed{\frac{x^{n+1}}{n+1} \left(\ln x - \frac{1}{n+1} \right) + C}\end{aligned}$$

7.2.63 Evaluate $\int (\sin^{-1} x)^2 dx$. Hint: Use Integration by Parts and then substitution.

We first use Integration by Parts:

$$u = (\sin^{-1} x)^2 \quad v = 1$$

$$u' = 2 \sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}} \quad v = x$$

$$\therefore \int (\sin^{-1} x)^2 dx = x(\sin^{-1} x)^2 - \int 2 \sin^{-1} x \cdot \frac{x}{\sqrt{1-x^2}} dx$$

Now use substitution on the integral on the right with $u = \sin^{-1} x$:

$$\int 2 \sin^{-1} x \cdot \frac{x}{\sqrt{1-x^2}} dx = \int 2u \cdot \frac{x}{\sqrt{1-x^2}} \cdot (\sqrt{1-x^2}) du$$

So we still have an x , but if $u = \sin^{-1} x$
then $x = \sin u$. So

$$\int 2 \sin^{-1} x \cdot \frac{x}{\sqrt{1-x^2}} dx = \int 2u \sin u du$$

$$w = 2u \Rightarrow w' = 2, \quad v = \sin u \Rightarrow v = -\cos u$$

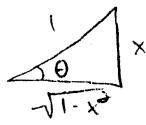
$$\therefore \int 2 \sin^{-1} x \cdot \frac{x}{\sqrt{1-x^2}} dx = -2u \cos u + \int 2 \cos u du$$

$$= -2u \cos u + 2 \sin u + C$$

Plugging this back into our original equation,
we get

$$\begin{aligned} \int (\sin^{-1} x)^2 dx &= x(\sin^{-1} x)^2 - (-2u \cos u + 2 \sin u) + C \\ &= x(\sin^{-1} x)^2 + 2 \sin^{-1} x \cos(\sin^{-1} x) - 2 \sin(\sin^{-1} x) + C \\ &= \boxed{x(\sin^{-1} x)^2 + 2 \sin^{-1} x (\sqrt{1-x^2}) - 2x + C} \end{aligned}$$

$$\theta = \sin^{-1} x$$



$$\cos \theta = \cos(\sin^{-1} x) = \frac{\sqrt{1-x^2}}{x}$$

7.2.66

Find $f(x)$, assuming that

$$\int f(x) e^x dx = f(x)e^x - \int x^{-1} e^x dx$$

Here we have used Integration by Parts with e^x , we must have that

$$u' = f'(x) = \frac{1}{x}$$

so that $f(x) = \int \frac{1}{x} = \ln x + C$ for some constant C .