

## MATH 20B HW5 SOLUTIONS

- Section 7.3: 1, 3, 10, 11, 22, 32, 33, 40, 47, 48, 52, 53
  - Section 7.5: 3, 4, 8, 11
  - Supplement 3.2: 1, 2, 4, 6, 8
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In Exercises 7.3.1 and 7.3.3, use the method for odd powers to evaluate the integral.

7.3.1

$$\int \cos^3 x \, dx$$

This is analogous to Example 1. So we use the identity  $\cos^2 x = 1 - \sin^2 x$  to write

$$\begin{aligned} \int \cos^3 x \, dx &= \int \cos x \cdot \cos^2 x \, dx \\ &= \int \cos x (1 - \sin^2 x) \, dx \end{aligned}$$

Using the substitution  $u = \sin x$  we have

$$\begin{aligned} \int \cos^3 x \, dx &= \int (1 - u^2) \, du \\ &= u - \frac{u^3}{3} + C \\ &= \boxed{\sin x - \frac{\sin^3 x}{3} + C} \end{aligned}$$

7.3.3

$$\int \sin^3 \theta \cos^2 \theta d\theta$$

This is analogous to Example 2, so we pick out the odd-powered trig function and write

$$\sin^3 \theta = \sin \theta \cdot \sin^2 \theta = \sin \theta (1 - \cos^2 \theta)$$

so that

$$\begin{aligned} \int \sin^3 \theta \cos^2 \theta d\theta &= \int \sin \theta (1 - \cos^2 \theta) \cos^2 \theta d\theta \\ &= \int \sin \theta (\cos^2 \theta - \cos^4 \theta) d\theta \end{aligned}$$

Letting  $u = \cos \theta$  we get  $du = -\sin \theta d\theta$ , so

$$\begin{aligned} \int \sin^3 \theta \cos^2 \theta d\theta &= \int -(u^2 - u^4) du \\ &= \int (u^4 - u^2) du \\ &= \frac{u^5}{5} - \frac{u^3}{3} + C \\ &= \boxed{\frac{\cos^5 \theta}{5} - \frac{\cos^3 \theta}{3} + C} \end{aligned}$$

In Exercises 7.3.10 and 7.3.11, evaluate the integrals using methods employed in Examples 3 and 4.

7.3.10

$$\int \cos^2 \theta \sin^2 \theta d\theta$$

We can choose  $\cos^2 \theta$  or  $\sin^2 \theta$ , but let's choose the former and set  $\cos^2 \theta = 1 - \sin^2 \theta$ :

$$\begin{aligned}\int \cos^2 \theta \sin^2 \theta d\theta &= \int (1 - \sin^2 \theta) \sin^2 \theta d\theta \\ &= \int \sin^2 \theta d\theta - \int \sin^4 \theta d\theta\end{aligned}$$

We apply a reduction formula for  $\int \sin^4 \theta d\theta$  to get

$$\int \sin^4 \theta d\theta = -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 \theta d\theta$$

so that plugging this in above we obtain

$$\begin{aligned}\int \cos^2 \theta \sin^2 \theta d\theta &= \int \sin^2 \theta d\theta - \left[ -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 \theta d\theta \right] \\ &= \frac{1}{4} \int \sin^2 \theta d\theta + \frac{1}{4} \sin^3 x \cos x\end{aligned}$$

We need a final reduction for  $\int \sin^2 \theta d\theta$ :

$$\begin{aligned}\int \sin^2 \theta d\theta &= -\frac{1}{2} \sin \theta \cos \theta + \frac{1}{2} \int \sin^0 \theta d\theta \\ &= -\frac{1}{2} \sin \theta \cos \theta + \frac{1}{2} \theta + C\end{aligned}$$

$$\begin{aligned}\therefore \int \cos^2 \theta \sin^2 \theta d\theta &= \frac{1}{4} \left[ -\frac{1}{2} \sin \theta \cos \theta + \frac{1}{2} \theta \right] + \frac{1}{4} \sin^3 x \cos x + C \\ &= \boxed{\frac{\theta}{8} - \frac{\sin \theta \cos \theta}{8} + \frac{\sin^3 x \cos x}{4} + C}\end{aligned}$$

7.3.11

$$\int \sin^4 x \cos^2 x dx$$

Here we have  $\cos^2 x = 1 - \sin^2 x$ , so

$$\int \sin^4 x \cos^2 x dx = \int \sin^4 x (1 - \sin^2 x) dx$$

$$= \int (\sin^4 x - \sin^6 x) dx = \int \sin^4 x dx - \int \sin^6 x dx$$

Our reduction formula for  $\int \sin^6 x dx$  gives

$$\int \sin^6 x dx = -\frac{1}{6} \sin^5 x \cos x + \frac{5}{6} \int \sin^4 x dx$$

$$\begin{aligned}\therefore \int \sin^4 x \cos^2 x dx &= \int \sin^4 x dx - \left[ -\frac{1}{6} \sin^5 x \cos x + \frac{5}{6} \int \sin^4 x dx \right] \\ &= \frac{1}{6} \int \sin^4 x dx + \frac{1}{6} \sin^5 x \cos x\end{aligned}$$

The reduction formula for  $\int \sin^4 x dx$  gives

$$\begin{aligned}\int \sin^4 x \cos^2 x dx &= \frac{1}{6} \left[ -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x dx \right] + \frac{1}{6} \sin^5 x \cos x \\ &= \frac{1}{8} \int \sin^2 x dx - \frac{1}{24} \sin^3 x \cos x + \frac{1}{6} \sin^5 x \cos x\end{aligned}$$

and the reduction for  $\int \sin^2 x dx$  gives

$$\begin{aligned}\int \sin^4 \cos^2 x dx &= \frac{1}{8} \left[ \frac{x}{2} - \frac{\sin 2x}{4} \right] - \frac{1}{24} \sin^3 x \cos x + \frac{1}{6} \sin^5 x \cos x + C \\ &= \boxed{\frac{x}{16} - \frac{\sin 2x}{32} - \frac{\sin^3 x \cos x}{24} + \frac{\sin^5 x \cos x}{6} + C}\end{aligned}$$

In the following exercises from §7.3, use the techniques and reduction formulas necessary to evaluate the integral.

7.3.22

$$\int \cos^3 2x \sin 2x \, dx$$

$$u = \cos 2x \Rightarrow du = -2 \sin 2x \, dx$$

$$\begin{aligned} \int \cos^3 2x \sin 2x \, dx &= \int -\frac{1}{2} u^3 du \\ &= -\frac{u^4}{8} + C \\ &= \boxed{-\frac{\cos^4 2x}{8} + C} \end{aligned}$$

7.3.32

$$\int \tan^3 \theta \sec^3 \theta \, d\theta$$

See the table on p. 436, which gives that we use  $\tan^2 \theta = \sec^2 \theta - 1$  to obtain

$$\begin{aligned} \int \tan^3 \theta \sec^3 \theta \, d\theta &= \int (\sec^2 \theta - 1) \tan \theta \sec^3 \theta \, d\theta \\ &= \int (\sec^2 \theta - 1) \tan \theta \sec \theta (\sec^2 \theta) \, d\theta \\ &= \int (\sec^4 \theta - \sec^2 \theta) \tan \theta \sec \theta \, d\theta \end{aligned}$$

Now we use u-substitution with  $u = \sec \theta \Rightarrow du = \sec \theta \tan \theta \, d\theta$

$$\begin{aligned} \int \tan^3 \theta \sec^3 \theta \, d\theta &= \int (u^4 - u^2) \, du \\ &= \frac{u^5}{5} - \frac{u^3}{3} + C \\ &= \boxed{\frac{\sec^5 \theta}{5} - \frac{\sec^3 \theta}{3} + C} \end{aligned}$$

7.3.33

$$\int \tan^5 x \sec^4 x dx$$

We proceed as in the previous exercise:

$$\begin{aligned}
 \int \tan^5 x \sec^4 x dx &= \int \tan^4 x \cdot \tan x \cdot \sec x \cdot \sec^3 x dx \\
 &= \int (\tan^2 x)^2 \cdot \sec^3 x \cdot (\sec x \tan x) dx \\
 &= \int (1 - \sec^2 x)^2 \cdot \sec^3 x \cdot (\sec x \tan x) dx \\
 &= \int (1 - 2\sec^2 x + \sec^4 x) \cdot \sec^3 x \cdot (\sec x \tan x) dx \\
 &= \int (\sec^3 x - 2\sec^5 x + \sec^7 x) \cdot (\sec x \tan x) dx
 \end{aligned}$$

$$u = \sec x \Rightarrow du = \sec x \tan x dx$$

$$\begin{aligned}
 \int \tan^5 x \sec^4 x dx &= \int (u^3 - 2u^5 + u^7) du \\
 &= \frac{u^4}{4} - \frac{u^6}{3} + \frac{u^8}{8} + C \\
 &= \boxed{\frac{\sec^4 x}{4} - \frac{\sec^6 x}{3} + \frac{\sec^8 x}{8} + C}
 \end{aligned}$$

7.3.40

$$\int \cos 4x \cos 6x dx = \int \cos 6x \cos 4x dx$$

Apply equation (27) in the table with  $m=6, n=4$

$$\int \cos 4x \cos 6x dx = \boxed{\frac{\sin 2x}{4} + \frac{\sin 10x}{20} + C}$$

7.3.47

$$\int_0^{\pi/3} \sin^3 x dx$$

We use a reduction formula:

$$\begin{aligned}\int_0^{\pi/3} \sin^3 x dx &= -\left. \frac{\sin^2 x \cos x}{3} \right|_0^{\pi/3} + \frac{2}{3} \int_0^{\pi/3} \sin x dx \\&= -\left. \frac{\sin^2 x \cos x}{3} \right|_0^{\pi/3} - \left. \frac{2}{3} \cos x \right|_0^{\pi/3} \\&= -\frac{1}{3} \left[ \left( \frac{\sqrt{3}}{2} \right)^2 \cdot \left( \frac{1}{2} \right) \right] - \frac{2}{3} \left[ \frac{1}{2} - 1 \right] \\&= -\frac{1}{3} \left( \frac{3}{8} \right) - \frac{2}{3} \left( -\frac{1}{2} \right) \\&= -\frac{1}{8} + \frac{1}{3} \\&= \boxed{\frac{5}{24}}\end{aligned}$$

7.3.48

$$\begin{aligned}
 \int_0^{\pi/4} \frac{dx}{\cos x} &= \int_0^{\pi/4} \sec x \, dx \\
 &= \ln |\sec x + \tan x| \Big|_0^{\pi/4} \quad (\text{by Table, eqn. (21)}) \\
 &= \ln \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| - \ln |\sec 0 + \tan 0| \\
 &= \ln |\sqrt{2} + 1| - \ln |1 + 0| \\
 &= \ln (\sqrt{2} + 1) - \ln (1) \\
 &= \boxed{\ln (\sqrt{2} + 1)}
 \end{aligned}$$

7.3.52

$$\int_{-\pi/4}^{\pi/4} \sec^4 x \, dx$$

By a reduction formula,

$$\begin{aligned}
 \int_{-\pi/4}^{\pi/4} \sec^4 x \, dx &= \frac{\tan x \sec^2 x}{3} \Big|_{-\pi/4}^{\pi/4} + \frac{2}{3} \int_{-\pi/4}^{\pi/4} \sec^2 x \, dx \\
 &= \frac{\tan x \sec^2 x}{3} \Big|_{-\pi/4}^{\pi/4} + \frac{2}{3} [\tan x] \Big|_{-\pi/4}^{\pi/4} \\
 &= \frac{1}{3} \left[ \tan x \sec^2 x + 2 \tan x \right] \Big|_{-\pi/4}^{\pi/4} \\
 &= \frac{1}{3} \left[ \left( 1 \cdot (\sqrt{2})^2 + 2 \cdot 1 \right) - \left( (-1)(-\sqrt{2})^2 + 2 \cdot (-1) \right) \right] \\
 &= \frac{1}{3} \left[ (2+2) - (-2-2) \right] \\
 &= \frac{1}{3} [4+4] \\
 &= \boxed{\frac{8}{3}}
 \end{aligned}$$

7.3.53

$$\int_0^{\pi} \sin 3x \cos 4x \, dx$$

We use equation (26) from the table with  $m=3, n=4$ :

$$\begin{aligned}\int_0^{\pi} \sin 3x \cos 4x \, dx &= \left[ -\frac{\cos(-x)}{2} - \frac{\cos(7x)}{14} \right]_0^{\pi} \\ &= \left[ \frac{\cos(-x)}{2} - \frac{\cos(7x)}{14} \right]_0^{\pi} \\ &= \left[ \left( \frac{\cos -\pi}{2} - \frac{\cos 7\pi}{14} \right) - \left( \frac{\cos 0}{2} - \frac{\cos 0}{14} \right) \right] \\ &= \left[ \left( \frac{-1}{2} + \frac{1}{14} \right) - \left( \frac{1}{2} - \frac{1}{14} \right) \right] \\ &= -1 + \frac{1}{7} = \boxed{-\frac{6}{7}}\end{aligned}$$

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In the exercises from section 7.5, evaluate the integral.

7.5.3

$$\int x \sinh(x^2+1) \, dx$$

$$u = x^2 + 1 \Rightarrow du = 2x \, dx$$

$$\begin{aligned}\therefore \int x \sinh(x^2+1) \, dx &= \int \frac{1}{2} \sinh(u) du \\ &= \frac{1}{2} \cosh(u) + C \\ &= \boxed{\frac{1}{2} \cosh(x^2+1) + C}\end{aligned}$$

7.5.4

$$\int \sinh^2 x \cosh x dx$$

$$u = \sinh x \Rightarrow du = \cosh x dx$$

$$\therefore \int \sinh^2 x \cosh x dx = \int u^2 du$$

$$= \frac{u^3}{3} + C$$

$$= \boxed{\frac{\sinh^3 x}{3} + C}$$

7.5.8

$$\int \frac{\cosh x}{3\sinh x + 4} dx$$

$$u = 3\sinh x + 4 \Rightarrow du = 3\cosh x dx$$

$$\therefore \int \frac{\cosh x}{3\sinh x + 4} dx = \int \frac{1}{3u} du$$

$$= \frac{1}{3} \ln |u| + C$$

$$= \boxed{\frac{1}{3} \ln |3\sinh x + 4| + C}$$

7.5.11

$$\int \frac{\cosh x}{\sinh x} dx$$

$$u = \sinh x \Rightarrow du = \cosh x dx$$

$$\therefore \int \frac{\cosh x}{\sinh x} dx = \int \frac{du}{u}$$

$$= \ln |u| + C$$

$$= \boxed{\ln |\sinh x| + C}$$

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Compute the following integrals using complex exponentials.

Supp. 3.2.1

$$\begin{aligned}
 \int_{-\pi}^{\pi} 7 \sin(5x) \cos(3x) dx &= \int_{-\pi}^{\pi} 7 \cdot \left( \frac{e^{5ix} - e^{-5ix}}{2i} \right) \left( \frac{e^{3ix} + e^{-3ix}}{2} \right) dx \\
 &= \int_{-\pi}^{\pi} \frac{7}{4i} \left( e^{8ix} + e^{2ix} - e^{-2ix} - e^{-8ix} \right) dx \\
 &= \int_{-\pi}^{\pi} \frac{7}{4i} \left[ \left( e^{8ix} - e^{-8ix} \right) + \left( e^{2ix} - e^{-2ix} \right) \right] dx \\
 &= \int_{-\pi}^{\pi} \frac{7}{4i} \cdot 2i \left( \frac{e^{8ix} - e^{-8ix}}{2i} + \frac{e^{2ix} - e^{-2ix}}{2i} \right) dx \\
 &= \int_{-\pi}^{\pi} \frac{7}{2} \left( \sin 8x + \sin 2x \right) dx \\
 &= \left[ -\frac{7}{16} \cos 8x - \frac{7}{4} \cos 2x \right]_{-\pi}^{\pi} \\
 &= \left( -\frac{7}{16} - \frac{7}{4} \right) - \left( -\frac{7}{16} - \frac{7}{4} \right) \\
 &= \boxed{0}
 \end{aligned}$$

Supp. 3.2.2

$$\begin{aligned}
 \int e^{ix} \cos(2x) dx &= \int e^{ix} \left( \frac{e^{2ix} + e^{-2ix}}{2} \right) dx \\
 &= \int \frac{1}{2} \left( e^{9ix} + e^{5ix} \right) dx \\
 &= \frac{1}{18i} e^{9ix} + \frac{1}{5i} e^{5ix} + C
 \end{aligned}$$

(because recall

that  $\frac{d}{dx} [e^{(a+bi)x}] = (a+bi) \cdot e^{(a+bi)x}$ )

$$= \boxed{\frac{-i}{18} e^{9ix} - \frac{i}{10} e^{5ix} + C}$$

Sup. 3.2.4.  $\int e^{-7x} \sin(2x) dx = \int e^{-7x} \left( \frac{e^{2ix} - e^{-2ix}}{2i} \right) dx$

$$= \int \frac{1}{2i} \left( e^{(-7+2i)x} - e^{(-7-2i)x} \right) dx$$

$$= \frac{1}{2i} \left( \frac{e^{(-7+2i)x}}{-7+2i} - \frac{e^{(-7-2i)x}}{-7-2i} \right) + C$$

$$= \frac{1}{2i} \left( \frac{(-7-2i)e^{(-7+2i)x}}{(-7+2i)(-7-2i)} - \frac{(-7+2i)e^{(-7-2i)x}}{(-7-2i)(-7+2i)} \right) + C$$

$$= \frac{1}{2i} \left[ \frac{(-7-2i)e^{(-7+2i)x}}{53} - \frac{(-7+2i)e^{(-7-2i)x}}{53} \right] + C$$

$$= \frac{1}{106i} \left[ -7e^{(-7+2i)x} - 2ie^{(-7+2i)x} + 7e^{(-7-2i)x} - 2ie^{(-7-2i)x} \right] + C$$

$$= \frac{1}{106i} \left[ -7e^{-7x} e^{2ix} - 2ie^{-7x} e^{2ix} + 7e^{-7x} e^{-2ix} - 2ie^{-7x} e^{-2ix} \right] + C$$

$$= \frac{e^{-7x}}{106i} \left[ -7e^{2ix} - 2ie^{2ix} + 7e^{-2ix} - 2ie^{-2ix} \right] + C$$

$$= \frac{e^{-7x}}{106i} \left[ (-7e^{2ix} + 7e^{-2ix}) + (-2ie^{2ix} - 2ie^{-2ix}) \right] + C$$

$$= \frac{e^{-7x}}{106i} \left[ -7(e^{2ix} + e^{-2ix}) - 2i(e^{2ix} + e^{-2ix}) \right] + C$$

$$= \frac{e^{-7x}}{106i} \left[ -7-2i \left( \frac{e^{2ix} - e^{-2ix}}{2i} \right) + 2i \cdot 2 \left( \frac{e^{2ix} + e^{-2ix}}{2} \right) \right] + C$$

$$= \frac{e^{-7x}}{106i} \left[ -14i \sin 2x - 4i \cos 2x \right] + C$$

$$= \boxed{\frac{e^{-7x}}{53} \left[ -7 \sin 2x - 2 \cos 2x \right] + C}$$

Supp. 3.2.6

$$\begin{aligned} \int \cos^3 x \cos(7x) dx &= \int \left( \frac{e^{ix} + e^{-ix}}{2} \right)^3 \left( \frac{e^{7ix} + e^{-7ix}}{2} \right) dx \\ &= \int \frac{1}{16} \left( e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-3ix} \right) \left( e^{7ix} + e^{-7ix} \right) dx \\ &= \int \frac{1}{16} \left( e^{10ix} + 3e^{8ix} + 3e^{6ix} + e^{4ix} + e^{-4ix} + 3e^{-6ix} + 3e^{-8ix} + e^{-10ix} \right) dx \\ &= \frac{1}{16} \int \left[ \left( e^{10ix} + e^{-10ix} \right) + 3 \left( e^{8ix} + e^{-8ix} \right) + 3 \left( e^{6ix} + e^{-6ix} \right) + \left( e^{4ix} + e^{-4ix} \right) \right] dx \\ &= \frac{1}{16} \int \left[ 2 \left( \frac{e^{10ix} + e^{-10ix}}{2} \right) + 3 \cdot 2 \left( \frac{e^{8ix} + e^{-8ix}}{2} \right) + 3 \cdot 2 \left( \frac{e^{6ix} + e^{-6ix}}{2} \right) + 2 \left( \frac{e^{4ix} + e^{-4ix}}{2} \right) \right] dx \\ &= \frac{1}{16} \int \left[ 2\cos(10x) + 6\cos(8x) + 6\cos(6x) + 2\cos(4x) \right] dx \\ &= \frac{1}{16} \left[ \frac{2\sin(10x)}{10} + \frac{6\sin(8x)}{8} + \frac{6\sin(6x)}{6} + \frac{2\sin(4x)}{4} \right] + C \\ &= \frac{1}{16} \left( \frac{\sin(10x)}{5} + \frac{3\sin(8x)}{4} + \sin(6x) + \frac{\sin(4x)}{2} \right) + C \\ &= \boxed{\frac{\sin(10x)}{80} + \frac{3\sin(8x)}{64} + \frac{\sin(6x)}{16} + \frac{\sin(4x)}{32} + C} \end{aligned}$$

Supp. 3.2.8

$$\begin{aligned}\int x \cos^3(x) dx &= \int x \cdot \frac{1}{8} \left( e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-3ix} \right) dx \\&= \frac{1}{8} \int x \left[ \left( e^{3ix} + e^{-3ix} \right) + 3 \left( e^{ix} + e^{-ix} \right) \right] dx \\&= \frac{1}{8} \int 2x \left[ \left( \frac{e^{3ix} + e^{-3ix}}{2} \right) + 3 \left( \frac{e^{ix} + e^{-ix}}{2} \right) \right] dx \\&= \frac{1}{8} \int 2x \left( \cos 3x + 3 \cos x \right) dx \\&= \int \frac{x}{4} \cos 3x dx + \int \frac{3}{4} x \cos x dx\end{aligned}$$

$$u_1 = \frac{x}{4}, \quad v_1' = \cos 3x \quad u_2 = \frac{3x}{4}, \quad v_2' = \cos x$$

$$u_1' = \frac{1}{4}, \quad v_1 = \frac{\sin 3x}{3} \quad u_2' = \frac{3}{4}, \quad v_2 = \sin x$$

$$\begin{aligned}\therefore \int x \cos^3(x) dx &= \frac{x \sin 3x}{12} - \frac{1}{12} \int \sin 3x dx + \frac{3x \sin x}{4} - \int \frac{3}{4} \sin x dx \\&= \boxed{\frac{x \sin 3x}{12} + \frac{\cos 3x}{36} + \frac{3x \sin x}{4} + \frac{3 \cos x}{4} + C}\end{aligned}$$