

MATH 20B HW6 SOLUTIONS

- Section 7.6: 6, 8, 12, 22, 26, 36, 46, 67
- Supplement 4.4: 1, 2, 3
- Supplement 5.6: 1, 2, 3, 4
- Section 7.7: 8, 15, 16, 28, 35, 42, 44, 45, 67, 70, 72

In Exercises 7.6.6 and 7.6.8, use long division to write $f(x)$ as the sum of a polynomial and a proper rational function. Then calculate $\int f(x) dx$.

7.6.6

$$f(x) = \frac{x^2 + 2}{x + 3}$$

$$x + 3 \overline{) x^2 + 2}$$

$$\begin{array}{r} x - 3 + \frac{11}{x + 3} \\ \underline{x^2 + 3x} \\ -3x + 2 \\ \underline{-3x - 9} \\ 11 \end{array}$$

$$\therefore f(x) = x - 3 + \frac{11}{x + 3}$$

$$\int f(x) dx = \int \left(x - 3 + \frac{11}{x + 3} \right) dx$$

$$= \boxed{\frac{x^2}{2} - 3x + 11 \ln |x + 3| + C}$$

7.6.8

$$f(x) = \frac{x^3 - 1}{x^2 - x}$$

$$x^2 - x \overline{) \frac{x^3 - 1}{x^3 - x^2}} \quad x + 1 + \frac{x-1}{x^2-x}$$

$$\frac{x^2 - 1}{x^2 - x}$$

$$\frac{x-1}{x-1}$$

$$\therefore f(x) = x + 1 + \frac{x-1}{x^2-x}$$

$$= x + 1 + \frac{x-1}{x(x-1)}$$

$$= x + 1 + \frac{1}{x}$$

$$\int f(x) dx = \int \left(x + 1 + \frac{1}{x} \right) dx$$

$$= \boxed{\frac{x^2}{2} + x + \ln|x| + C}$$

In Exercises 7.6. 12, 22, 26, 36, and 46, evaluate the integral.

7.6.12

$$\int \frac{(3x+5)}{x^2-4x-5} dx = \int \left(\frac{A}{x-5} + \frac{B}{x+1} \right) dx$$
$$= \int \frac{A(x+1) + B(x-5)}{x^2-4x-5} dx$$

$$\therefore 3x+5 = A(x+1) + B(x-5)$$

$$= x(A+B) + A - 5B$$

$$A+B=3 \Rightarrow A=3-B \quad A-5B=5$$

$$(3-B) - 5B = 5$$

$$3-6B=5$$

$$6B = -2 \Rightarrow B = -\frac{1}{3} \Rightarrow A = \frac{10}{3}$$

$$\therefore \int \frac{3x+5}{x^2-4x-5} dx = \int \left(\frac{10/3}{x-5} - \frac{1/3}{x+1} \right) dx$$

$$= \boxed{\frac{10}{3} \ln|x-5| - \frac{1}{3} \ln|x+1| + C}$$

7.6.22

$$\int \frac{(x^2+x+3)}{(x-1)^3} dx = \int \left[\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} \right] dx$$

$$= \int \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3} dx$$

$$x^2+x+3 = A(x-1)^2 + B(x-1) + C$$

$$= A(x^2 - 2x + 1) + B(x-1) + C$$

$$= Ax^2 - 2Ax + A + Bx - B + C$$

$$= Ax^2 + (B-2A)x + A-B+C$$

$$A=1, \quad B-2A=1 \Rightarrow B=3, \quad A-B+C=3 \Rightarrow C=5$$

$$\therefore \int \frac{x^2+x+3}{(x-1)^3} dx = \int \left[\frac{1}{x-1} + \frac{3}{(x-1)^2} + \frac{5}{(x-1)^3} \right] dx$$

$$= \boxed{\ln|x-1| - \frac{3}{x-1} - \frac{5}{2(x-1)^2} + C}$$

7.6.26

$$\int \frac{dx}{x(x-1)^3} = \int \left[\frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} \right] dx$$

$$1 = A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx$$

$$= (Ax^3 - 3Ax^2 + 3Ax - A) + (Bx^3 - 2Bx^2 + Bx) + (Cx^2 - Cx) + Dx$$

$$= x^3(A+B) + x^2(-3A-2B+C) + x(3A+B-C+D) - A$$

$$\therefore A=-1, \quad A+B=0 \Rightarrow B=1, \quad -3A-2B+C=0 \Rightarrow C=-1,$$

$$3A+B-C+D=0 \Rightarrow D=1$$

$$\therefore \int \frac{dx}{x(x-1)^3} = \int \left[\frac{-1}{x} + \frac{1}{x-1} - \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3} \right] dx$$

$$= \boxed{-\ln|x| + \ln|x-1| + \frac{1}{x-1} - \frac{1}{2(x-1)^2} + C}$$

7.6.36

$$\int \frac{dx}{x(x^2+25)} = \int \left[\frac{A}{x} + \frac{Bx+C}{x^2+25} \right] dx$$

$$1 = A(x^2+25) + x(Bx+C)$$

$$1 = Ax^2 + 25A + Bx^2 + Cx$$

$$= x^2(A+B) + Cx + 25A$$

$$\therefore 25A = 1 \Rightarrow A = \frac{1}{25}, \quad C = 0, \quad A+B=1 \Rightarrow B = -\frac{1}{25}$$

$$\int \frac{dx}{x(x^2+25)} = \int \left(\frac{1/25}{x} - \frac{(1/25)x}{x^2+25} \right) dx$$

$$= \boxed{\frac{1}{25} \ln|x| - \frac{1}{50} \ln(x^2+25) + C}$$

7.6.46

$$\int \frac{dx}{x^4-1} = \int \left(\frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} \right) dx$$

$$1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x+1)(x-1)$$

$$= A(x^3 - x^2 + x - 1) + B(x^3 + x^2 + x + 1) + (Cx^3 + Dx^2 - Cx - D)$$

$$= x^3(A+B+C) + x^2(-A+B+D) + x(A+B-C) - A+B-D$$

$$-A+B-D=1$$

$$A+B+C=0$$

$$-A+B+D=0$$

$$-A+B-C=0$$

$$-2D=1 \Rightarrow D=-\frac{1}{2}$$

$$2C=0 \Rightarrow C=0$$

$$\rightarrow A+B+C=0 \Rightarrow A=-B, \quad -A+B-D=1$$

$$\therefore -A+B = \frac{1}{2}$$

$$2B = \frac{1}{2} \Rightarrow B = \frac{1}{4} \Rightarrow A = -\frac{1}{4}$$

$$\begin{aligned} \therefore \int \frac{dx}{x^4-1} &= \int \left[\frac{-1/4}{x+1} + \frac{1/4}{x-1} - \frac{1/2}{x^2+1} \right] dx \\ &= \boxed{-\frac{1}{4} \ln|x+1| + \frac{1}{4} \ln|x-1| - \frac{1}{2} \tan^{-1}(x) + C} \end{aligned}$$

7.6.67

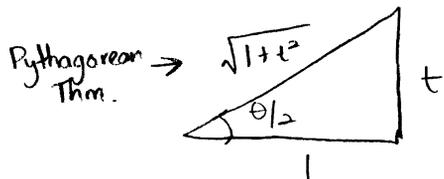
Show that the substitution $\theta = 2 \tan^{-1} t$ yields the formulas

$$\cos \theta = \frac{1-t^2}{1+t^2}, \quad \sin \theta = \frac{2t}{1+t^2}, \quad d\theta = \frac{2dt}{1+t^2}$$

Use it to evaluate

$$\int \frac{d\theta}{\cos \theta + (3/4) \sin \theta}$$

We can see that $d\theta/dt = 2/(1+t^2)$. Also, if $\theta = 2 \tan^{-1} t$ then $\theta/2 = \tan^{-1} t$ so that $\tan \theta/2 = t = t/1$. Hence we can construct the triangle



$$\therefore \cos \frac{\theta}{2} = \frac{\text{adj.}}{\text{hyp.}} = \frac{1}{\sqrt{t^2+1}}, \quad \sin \frac{\theta}{2} = \frac{\text{opp.}}{\text{hyp.}} = \frac{t}{\sqrt{t^2+1}}$$

By double-angle formulas,

$$\begin{aligned} \cos \theta &= \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} & \sin \theta &= 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \\ &= \frac{1}{t^2+1} - \frac{t^2}{t^2+1} & &= 2 \cdot \frac{t}{\sqrt{t^2+1}} \cdot \frac{1}{\sqrt{t^2+1}} \\ &= \frac{1-t^2}{t^2+1} \quad \checkmark & &= \frac{2t}{t^2+1} \quad \checkmark \end{aligned}$$

$$\begin{aligned} \int \frac{d\theta}{\cos \theta + (3/4) \sin \theta} &= \int \frac{2dt/(1+t^2)}{(1-t^2)/(t^2+1) + (3/4)(2t/(t^2+1))} \\ &= \int \frac{2 \sqrt{1+t^2}}{(4-4t^2+6t)/4(t^2+1)} dt \\ &= \int \frac{8}{-4t^2+6t+4} dt \end{aligned}$$

$$\begin{aligned} \int \frac{d\theta}{\cos\theta + (\frac{3}{4})\sin\theta} &= \int \frac{-4}{2t^2 - 3t - 2} dt \\ &= \int \left[\frac{A}{(2t+1)} + \frac{B}{(t-2)} \right] dt \\ &= \int \frac{A(t-2) + B(2t+1)}{2t^2 - 3t - 2} dt \end{aligned}$$

$$\therefore -4 = A(t-2) + B(2t+1)$$

$$= At - 2A + 2Bt + B$$

$$= t(A+2B) - 2A + B$$

$$\therefore A+2B=0 \Rightarrow A = -2B$$

$$-2A + B = -4$$

$$-2(-2B) + B = -4$$

$$5B = -4 \Rightarrow B = \frac{-4}{5} \Rightarrow A = -2\left(\frac{-4}{5}\right) = \frac{8}{5}$$

$$\begin{aligned} \int \frac{d\theta}{\cos\theta + (\frac{3}{4})\sin\theta} &= \int \left[\frac{8/5}{2t+1} - \frac{4/5}{t-2} \right] dt \\ &= \frac{8}{10} \ln|2t+1| - \frac{4}{5} \ln|t-2| + C \\ &= \frac{4}{5} \ln|2t+1| - \frac{4}{5} \ln|t-2| + C \end{aligned}$$

Using our substitution $t = \tan(\frac{\theta}{2})$, we have

$$\boxed{\int \frac{d\theta}{\cos\theta + (\frac{3}{4})\sin\theta} = \frac{4}{5} \ln\left|2\tan\left(\frac{\theta}{2}\right)+1\right| - \frac{4}{5} \ln\left|\tan\left(\frac{\theta}{2}\right)-2\right| + C}$$

The book has $|2 - \tan(\frac{\theta}{2})|$ here, but we are taking absolute values so either way works

Supp. 4.4.1 Expand $Q(x) = (x-2)(x-3)(x^2+1)$ in the form
 $Q(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$.

$$\begin{aligned} Q(x) &= (x-2)(x-3)(x^2+1) \\ &= (x^2-5x+6)(x^2+1) \\ &= x^4 + x^2 - 5x^3 - 5x + 6x^2 + 6 \\ &= \boxed{x^4 - 5x^3 + 7x^2 - 5x + 6} \end{aligned}$$

Supp. 4.4.2 Show that if P is a polynomial and $P(5) = 0$, then $\frac{P(x)}{x-5}$ is a polynomial.

By the Fundamental Thm. of Algebra, we know that
 $P(x) = c(x-5)^{m_1}(x-r_2)^{m_2} \dots (x-r_\ell)^{m_\ell} (x^2+b_1x+c_1)^{n_1} \dots (x^2+b_kx+c_k)^{n_k}$

Since $m_1 \geq 1$, dividing by $x-5$ leaves
 $\frac{P(x)}{x-5} = c(x-5)^{m_1-1}(x-r_2)^{m_2} \dots (x-r_\ell)^{m_\ell} (x^2+b_1x+c_1)^{n_1} \dots (x^2+b_kx+c_k)^{n_k}$
which is also a polynomial, as needed.

Supp. 4.4.3 (a) How many poles does the rational function

$$r(x) = \frac{3}{5x^3+x+6}$$

have? Does it have a "pole at ∞ "?

First note that $5x^3+x+6 = (x+1)(5x^2-5x+6)$, so $x=-1$ is a pole, as are the (non-real) solutions to $5x^2-5x+6=0$:

$$x = \frac{5 \pm \sqrt{25-120}}{10} = \frac{5 \pm i\sqrt{95}}{10}$$

At the same time, $\lim_{x \rightarrow \infty} |r(x)| = 0 \neq \infty$, so $r(x)$ does not have a pole at ∞ .

(b) What are the pole locations and their multiplicities for

$$r(x) = \frac{3-2x}{(x-2)(x^2+5x+7)}$$

$r(x)$ clearly has a pole of multiplicity 1 at $x=2$. To find the other poles, we solve $x^2+5x+7=0$ by the quadratic equation:

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{25-28}}{2} \\ &= \frac{-5 \pm i\sqrt{3}}{2} \end{aligned}$$

(Each of these poles has multiplicity 1).

Supp. 5.6.1

Find the partial fraction expansion of $\frac{2x+1}{(x-1)^2(x+2)}$.

$$\begin{aligned}\frac{2x+1}{(x-1)^2(x+2)} &= \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \\ &= \frac{A(x-1)^2 + B(x+2)(x-1) + C(x+2)}{(x-1)^2(x+2)}\end{aligned}$$

$$\begin{aligned}2x+1 &= A(x^2-2x+1) + B(x^2+x-2) + C(x+2) \\ &= x^2(A+B) + x(-2A+B+C) + A-2B+2C\end{aligned}$$

$$A+B=0 \Rightarrow A=-B$$

$$-2A+B+C=2$$

$$A-2B+2C=1$$

$$2B+B+C=2$$

$$-B-2B+2C=1$$

$$3B+C=2$$

$$-3B+2C=1$$

$$C=2-3B$$

$$\Rightarrow -3B+2(2-3B)=1$$

$$-3B+4-6B=1$$

$$-9B+4=1$$

$$9B=3 \Rightarrow B=\frac{1}{3}$$

$$\therefore C=1, A=-\frac{1}{3}$$

$$\therefore \boxed{\frac{2x+1}{(x-1)^2(x+2)} = \frac{-\frac{1}{3}}{x+2} + \frac{\frac{1}{3}}{x-1} + \frac{1}{(x-1)^2}}$$

* Note: I did this exercise according to the methods in the text, although the methods in the supplement can be used also.

Supp. 5.6.2

Given

$$f(x) = \frac{3}{(x-1)(x-2)^2}$$

What value of A makes $f(x) - \frac{A}{x-1}$ have its only pole located at 2?

If we have a partial fraction expansion

$$f(x) = \frac{B}{x-1} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$$

then A is just the value B , since if $A=B$ then

$$f(x) - \frac{A}{x-1} = \frac{C}{x-2} + \frac{D}{(x-2)^2}$$

which only has a pole at 2.

To determine the value of A , it suffices to find $f(x)(x-1)|_{x=1}$:

$$f(x)(x-1)|_{x=1} = \frac{3}{(1-2)^2} = \boxed{3}$$

Supp. 5.6.3 Find the partial fraction expansion of

$$\frac{x^3+2}{x(x^2+1)(x^2+4)}$$

We have two irreducible quadratics, so let's try partial fractions with the complex numbers:

$$\begin{aligned} \frac{x^3+2}{x(x^2+1)(x^2+4)} &= \frac{x^3+2}{x(x+i)(x-i)(x+2i)(x-2i)} \\ &= \frac{A}{x} + \frac{B}{x+i} + \frac{C}{x-i} + \frac{D}{x+2i} + \frac{E}{x-2i} \end{aligned}$$

We apply the methods of the section since we have poles of multiplicity 1:

$$A = \frac{x^3+2}{x(x^2+1)(x^2+4)} \Big|_{x=0} = \frac{0^3+2}{(0^2+1)(0^2+4)} = \frac{1}{2}$$

$$\begin{aligned} B &= \frac{x^3+2}{x(x^2+1)(x^2+4)} (x+i) \Big|_{x=-i} = \frac{(-i)^3+2}{-i(-i-i)((-i)^2+4)} = \frac{2+i}{-3i(-2i)} \\ &= \frac{2+i}{-6} \\ &= -\frac{1}{3} - \frac{1}{6}i \end{aligned}$$

$$\begin{aligned} C &= \frac{x^3+2}{x(x^2+1)(x^2+4)} (x-i) \Big|_{x=i} = \frac{i^3+2}{i(i+i)(i^2+4)} \\ &= \frac{2-i}{3i(2i)} \\ &= \frac{2-i}{-6} \\ &= -\frac{1}{3} + \frac{1}{6}i \end{aligned}$$

$$\begin{aligned} D &= \frac{x^3+2}{x(x^2+1)(x^2+4)} (x+2i) \Big|_{x=-2i} = \frac{(-2i)^3+2}{(-2i)((-2i)^2+1)(-2i-2i)} \\ &= \frac{2+8i}{(-2i)(-3)(-4i)} \\ &= \frac{2+8i}{24} \\ &= \frac{1}{12} + \frac{1}{3}i \end{aligned}$$

$$\begin{aligned}
 E &= \frac{x^3+2}{x(x^2+1)(x^2+4)} (x-2i) \Big|_{x=2i} = \frac{(2i)^3+2}{2i(2i)^2+1)(2i+2i)} \\
 &= \frac{2-8i}{2i(-3)(4i)} \\
 &= \frac{2-8i}{24} \\
 &= \frac{1}{12} - \frac{1}{3}i
 \end{aligned}$$

(Note that $C = \bar{B}$, $E = \bar{D}$ so we didn't need to find E and C' explicitly, but it's a good check)

$$\therefore \frac{x^3+2}{x(x^2+1)(x^2+4)} = \frac{1/2}{x} + \frac{(-1/3 - 1/6i)}{x+i} + \frac{(-1/3 + 1/6i)}{x-i} + \frac{(1/12 + 1/3i)}{x+2i} + \frac{(1/12 - 1/3i)}{x-2i}$$

This is perfectly acceptable in the complex world, but we need to return to real numbers:

$$\begin{aligned}
 \frac{-1/3 - 1/6i}{x+i} + \frac{-1/3 + 1/6i}{x-i} &= \frac{(-1/3 - 1/6i)(x-i) + (-1/3 + 1/6i)(x+i)}{x^2+1} \\
 &= \frac{x(-1/3 - 1/6i) + (-1/6 + 1/3i) + x(-1/3 + 1/6i) + (-1/6 - 1/3i)}{x^2+1} \\
 &= \frac{-2/3x - 1/3}{x^2+1}
 \end{aligned}$$

$$\begin{aligned}
 \frac{1/12 + 1/3i}{x+2i} + \frac{1/12 - 1/3i}{x-2i} &= \frac{(1/12 + 1/3i)(x+2i) + (1/12 - 1/3i)(x-2i)}{x^2+4} \\
 &= \frac{x(1/12 + 1/3i) + (-2/3 + 1/6i) + x(1/12 - 1/3i) + (-2/3 - 1/6i)}{x^2+4} \\
 &= \frac{1/6x - 4/3}{x^2+4}
 \end{aligned}$$

Therefore

$$\boxed{\frac{x^3+2}{x(x^2+1)(x^2+4)} = \frac{1/2}{x} + \frac{-2/3x - 1/3}{x^2+1} + \frac{1/6x - 4/3}{x^2+4}}$$

Note: We could have used another method here of course, but it is nice to see it work with complex numbers. Both calculations are extremely tedious.

Find the partial fraction expansion of $\frac{x^3+2}{x(x^2-1)(x^2-4)}$

Here we have
$$\frac{x^3+2}{x(x^2-1)(x^2-4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} + \frac{D}{x-2} + \frac{E}{x+2}$$

and we can at least be slightly relieved that each pole is real of multiplicity 1, since we have

$$A = \frac{x^3+2}{x(x^2-1)(x^2-4)} \Big|_{x=0} = \frac{2}{(0^2-1)(0^2-4)} = \frac{4}{2} = \frac{1}{2}$$

$$B = \frac{x^3+2}{x(x^2-1)(x^2-4)} \Big|_{x=1} = \frac{1(1+1)(1^2-4)}{1^3+2} = \frac{-6}{3} = -\frac{2}{1}$$

$$C = \frac{x^3+2}{x(x^2-1)(x^2-4)} \Big|_{x=-1} = \frac{(-1)^3+2}{(-1)(-1-1)(1^2-4)} = \frac{-6}{-1} = \frac{6}{1}$$

$$D = \frac{x^3+2}{x(x^2-1)(x^2-4)} \Big|_{x=2} = \frac{2(2^2-1)(2+2)}{2^3+2} = \frac{10}{5} = \frac{2}{1}$$

$$E = \frac{x^3+2}{x(x^2-1)(x^2-4)} \Big|_{x=-2} = \frac{(-2)^3+2}{(-2)(-2-1)(2^2-4)} = \frac{-6}{-6} = \frac{4}{-1}$$

$$\frac{x^3+2}{x(x^2-1)(x^2-4)} = \frac{x}{1 \cdot 0} + \frac{-1-x}{1 \cdot 0} + \frac{1+x}{0 \cdot 1} + \frac{-x}{1 \cdot 0} + \frac{x+2}{0 \cdot 1}$$

In Exercises 7.7.8, 15, 16, 28, 35, 42, 44, and 45, determine whether the improper integral converges and, if so, evaluate it.

7.7.8

$$\begin{aligned} \int_{20}^{\infty} \frac{dt}{t} &= \lim_{b \rightarrow \infty} \int_{20}^b \frac{dt}{t} \\ &= \lim_{b \rightarrow \infty} \ln|t| \Big|_{20}^b \\ &= \lim_{b \rightarrow \infty} (\ln b - \ln 20) \\ &= \boxed{\text{Diverges}} \quad (\text{since } \ln b \rightarrow \infty \text{ as } b \rightarrow \infty) \end{aligned}$$

7.7.15

$$\begin{aligned} \int_{-3}^{\infty} \frac{dx}{(x+4)^{3/2}} &= \lim_{b \rightarrow \infty} \int_{-3}^b \frac{dx}{(x+4)^{3/2}} \\ &= \lim_{b \rightarrow \infty} \left. \frac{(x+4)^{-1/2}}{-1/2} \right|_{-3}^b \\ &= \lim_{b \rightarrow \infty} \left. \frac{-2}{\sqrt{x+4}} \right|_{-3}^b \\ &= \lim_{b \rightarrow \infty} \left[\frac{-2}{\sqrt{b+4}} + \frac{2}{\sqrt{-3+4}} \right] \\ &= \left(\lim_{b \rightarrow \infty} \frac{-2}{\sqrt{b+4}} \right) + \frac{2}{\sqrt{1}} \\ &= 0 + 2 = \boxed{2} \quad (\text{Converges}) \end{aligned}$$

7.7.16

$$\begin{aligned} \int_2^{\infty} e^{-2x} dx &= \lim_{b \rightarrow \infty} \int_2^b e^{-2x} dx \\ &= \lim_{b \rightarrow \infty} \left. \frac{e^{-2x}}{-2} \right|_2^b \\ &= \lim_{b \rightarrow \infty} \left(\frac{e^{-2b}}{-2} - \frac{e^{-4}}{-2} \right) \\ &= \lim_{b \rightarrow \infty} \left(\frac{-1}{2e^{2b}} \right) + \frac{1}{2e^4} = 0 + \frac{1}{2e^4} = \boxed{\frac{1}{2e^4}} \quad (\text{Converges}) \end{aligned}$$

7.7.28

$$\begin{aligned}
 \int_{-\infty}^0 x e^{-x^2} dx &= \lim_{a \rightarrow -\infty} \int_a^0 x e^{-x^2} dx \\
 &= \lim_{a \rightarrow -\infty} \left. \frac{-e^{-x^2}}{2} \right]_a^0 \\
 &= \lim_{a \rightarrow -\infty} \left(\frac{-e^0}{2} + \frac{e^{-a^2}}{2} \right) \\
 &= \frac{-1}{2} + \lim_{a \rightarrow -\infty} \left(\frac{1}{2e^{a^2}} \right) \\
 &= \boxed{\frac{-1}{2}} \text{ (Converges)}
 \end{aligned}$$

7.7.35

$$\begin{aligned}
 \int_1^{\infty} \frac{e^{\sqrt{x}}}{\sqrt{x}} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{e^{\sqrt{x}}}{\sqrt{x}} dx \\
 &= \lim_{b \rightarrow \infty} \left. 2e^{\sqrt{x}} \right]_1^b \\
 &= \lim_{b \rightarrow \infty} 2e^{\sqrt{b}} - 2e^{\sqrt{1}} \\
 &= \boxed{\text{Diverges}}
 \end{aligned}$$

7.7.42

$$\begin{aligned}
 \int_0^1 \ln x dx &= \lim_{a \rightarrow 0^+} \int_a^1 \ln x dx \\
 &= \lim_{a \rightarrow 0^+} \left[x \ln x - x \right]_a^1 \quad (\text{Integration by Parts}) \\
 &= \lim_{a \rightarrow 0^+} \left[(1 \ln 1 - 1) - (a \ln a - a) \right] \\
 &= \lim_{a \rightarrow 0^+} \left[-1 - a \ln a \right] \\
 &= -1 - \lim_{a \rightarrow 0^+} \frac{\ln a}{1/a} \\
 &= -1 + \lim_{a \rightarrow 0^+} \frac{1/a}{-1/a^2} \quad \left. \begin{array}{l} \text{L'Hôpital's} \\ \text{Rule} \end{array} \right\} \\
 &= -1 + \lim_{a \rightarrow 0^+} (-a) = -1 + 0 = \boxed{-1} \text{ (Converges)}
 \end{aligned}$$

7.7.44

$$\begin{aligned}
 \int_1^2 \frac{dx}{x \ln x} &= \lim_{a \rightarrow 1^+} \int_a^2 \frac{dx}{x \ln x} \\
 &= \lim_{a \rightarrow 1^+} \left[\ln(\ln x) \right]_a^2 \quad (\text{u-substitution, } u = \ln x) \\
 &= \lim_{a \rightarrow 1^+} \left[\ln(\ln 2) - \ln(\ln a) \right] \\
 &\boxed{\text{Diverges}} \quad (\text{because } \ln(\ln a) \rightarrow -\infty \text{ as } a \rightarrow 1^+)
 \end{aligned}$$

7.7.45

$$\int_0^1 \frac{\ln x}{x^2} dx = \lim_{a \rightarrow 0^+} \int_a^1 \frac{\ln x}{x^2} dx +$$

$$u = \ln x$$

$$v' = \frac{1}{x^2}$$

$$u' = \frac{1}{x}$$

$$v = \frac{-2}{x^3}$$

$$\begin{aligned}
 \int_0^1 \frac{\ln x}{x^2} dx &= \lim_{a \rightarrow 0^+} \left[\frac{-2 \ln x}{x^3} \Big|_a^1 + \int_a^1 \frac{2}{x^4} dx \right] + \\
 &= \lim_{a \rightarrow 0^+} \left[\frac{-2 \ln x}{x^3} - \frac{2}{3x^3} \right]_a^1 \\
 &= \lim_{a \rightarrow 0^+} \left[\left(\frac{-2 \ln 1}{1^3} - \frac{2}{3} \right) - \left(\frac{-2 \ln a}{a^3} - \frac{2}{3a^3} \right) \right] \\
 &= \lim_{a \rightarrow 0^+} \left[\left(0 - \frac{2}{3} \right) + \left(\frac{2 \ln a}{a^3} + \frac{2}{3a^3} \right) \right] \\
 &\boxed{\text{Diverges}}
 \end{aligned}$$

In the remaining exercises, use the Comparison Test to determine whether or not the integral converges.

7.7.67

$$\int_3^{\infty} \frac{dx}{\sqrt{x}-1}$$

Because $0 \leq \frac{1}{\sqrt{x}} \leq \frac{1}{\sqrt{x}-1}$ for all $x \geq 3$ and $\int_3^{\infty} \frac{1}{\sqrt{x}} dx$ diverges from the table in the section, we have that the given integral diverges as well.

7.7.70

$$\int_0^1 \frac{\sin x}{\sqrt{x}} dx$$

We know for all x that $-1 \leq \sin x \leq 1$, so it follows that

$$\frac{-1}{\sqrt{x}} \leq \frac{-|\sin x|}{\sqrt{x}} \leq \frac{\sin x}{\sqrt{x}} \leq \frac{|\sin x|}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$$

At the same time, we know from the text that $\int_0^1 \frac{1}{\sqrt{x}} dx$ converges, so it follows that $\int_0^1 \frac{|\sin x|}{\sqrt{x}} dx$ converges by the Comparison Test and hence $\int_0^1 \frac{\sin x}{\sqrt{x}} dx$ converges.

7.7.72

$$\int_1^{\infty} \frac{1}{x^4 + e^x} dx$$

Because $e^x > 0$ for $x \geq 1$, we have that

$$\frac{1}{x^4 + e^x} \leq \frac{1}{x^4}$$

for $x \geq 1$. But $\int_1^{\infty} \frac{1}{x^4} dx$ converges (again from the text), therefore so does the given integral.