

MATH 20B HW7 SOLUTIONS

- Section 10.1: 11, 14, 18, 23, 35, 43, 51

- Section 10.2: 9, 12, 15, 21, 25, 27, 32, 36

10.1.11 Find a formula for the n^{th} term of the following sequence:

(a) $\frac{1}{1}, -\frac{1}{8}, \frac{1}{27}, \dots$

We have an alternating sequence, so we'll have a factor of $(-1)^{n+1}$ since the first term is positive. Secondly, notice that $1=1^3$, $8=2^3$, and $27=3^3$, so we have

$$a_n = \frac{(-1)^{n+1}}{n^3}$$

(b) $\frac{2}{6}, \frac{3}{7}, \frac{4}{8}, \dots$

Observe that we start with $\frac{2}{6}$ and add one to the numerator and denominator each time. So we'll have

$$a_n = \frac{n+1}{n+5}$$

In Exercises 14, 18, and 23, use Theorem 1 to determine the limit of the sequence or state that the sequence diverges.

10.1.14

$$b_n = \frac{3n+1}{2n+4}$$

So let $f(x) = (3x+1)/(2x+4)$. By multiplying the numerator and denominator by $1/x$, we get

$$f(x) = \frac{3 + 1/x}{2 + 4/x}$$

(this only holds for $x \neq 0$ technically but we're taking $\lim_{x \rightarrow \infty}$ so we're fine). So

$$\lim_{x \rightarrow \infty} f(x) = \frac{3+0}{2+0} = \frac{3}{2}$$

Therefore by Theorem 1 we have

$$\boxed{\lim_{n \rightarrow \infty} b_n = \frac{3}{2}}$$

10.1.18

$$c_n = 4(2^n) \longrightarrow \boxed{\text{DIVERGES}} \quad (\{c_n\} \text{ is unbounded})$$

10.1.23

$$a_n = \frac{n}{\sqrt{n^3+1}}$$

Let $f(x) = x/\sqrt{x^3+1}$. We have

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^3+1}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x} \sqrt{x^3+1}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x^{1/2} + 1/x^2}} = 0 \end{aligned}$$

Therefore by Theorem 1

$$\boxed{\lim_{n \rightarrow \infty} a_n = 0}$$

10.1.35 Find

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

Take the natural logarithm of the limit:

$$\begin{aligned} \ln\left(\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n\right) &= \lim_{n \rightarrow \infty} \left(\ln\left(1 + \frac{1}{n}\right)^n\right) = \lim_{n \rightarrow \infty} \left(n \ln\left(1 + \frac{1}{n}\right)\right) \\ &= \lim_{n \rightarrow \infty} \frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}} = 1 \quad (\text{L'Hôpital}) \Rightarrow \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e' = \boxed{e} \end{aligned}$$

In Exercises 43 and 51, determine the limit of the sequence or show that the sequence diverges by using the appropriate Limit Laws or Theorems.

10.1.43

$$a_n = \frac{3n^2 + n + 2}{2n^2 - 3}$$

Let $f(x) = \frac{3x^2 + x + 2}{2x^2 - 3}$. We know that

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{3x^2 + x + 2}{2x^2 - 3} \\ &= \lim_{x \rightarrow \infty} \frac{3 + \frac{1}{x} + \frac{2}{x^2}}{2 - \frac{3}{x^2}} \\ &= \frac{3}{2} \end{aligned}$$

Therefore by Theorem 1

$$\lim_{n \rightarrow \infty} a_n = \boxed{\frac{3}{2}}$$

10.1.51

$$y_n = \frac{e^n + 3^n}{5^n}$$

Note that $y_n \geq 0$ for all n and also because $e < 3$, $e^n < 3^n$ for all n . Hence for all n

$$0 \leq \frac{e^n + 3^n}{5^n} \leq \frac{3^n + 3^n}{5^n} = \frac{2 \cdot 3^n}{5^n} = 2 \cdot \left(\frac{3}{5}\right)^n$$

So letting $x_n = 0$ for all n and $z_n = 2 \cdot \left(\frac{3}{5}\right)^n$, we have that

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = 0$$

But $x_n \leq y_n \leq z_n$ for all n , so by the Squeeze Theorem $\lim_{n \rightarrow \infty} y_n = 0$.

10.2.9

Calculate S_3 , S_4 , and S_5 and then find the sum of the telescoping series

$$S = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$$

$$\begin{aligned} S_3 &= \sum_{n=1}^3 \left(\frac{1}{n+1} - \frac{1}{n+2} \right) = \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) \\ &= \frac{1}{2} - \frac{1}{5} = \frac{3}{10} \end{aligned}$$

$$\begin{aligned} S_4 &= \sum_{n=1}^4 \left(\frac{1}{n+1} - \frac{1}{n+2} \right) = \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{6} \right) \\ &= \frac{1}{2} - \frac{1}{6} = \frac{1}{3} \end{aligned}$$

$$S_5 = \sum_{n=1}^5 \left(\frac{1}{n+1} - \frac{1}{n+2} \right) = \dots = \frac{1}{2} - \frac{1}{7} = \frac{5}{14}$$

$$\therefore S_N = \frac{1}{2} - \frac{1}{N+2}$$

$$\therefore S = \lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \left(\frac{1}{2} - \frac{1}{N+2} \right) = \boxed{\frac{1}{2}}$$

10.2.12

Find a formula for the partial sums S_N of $\sum_{n=1}^{\infty} (-1)^{n-1}$ and show that the series diverges.

Let's find a few partial sums:

$$S_1 = \sum_{n=1}^1 (-1)^{n-1} = 1$$

$$S_2 = \sum_{n=1}^2 (-1)^{n-1} = 1 - 1 = 0$$

$$S_3 = \sum_{n=1}^3 (-1)^{n-1} = 1 - 1 + 1 = 1$$

$$\therefore S_N = \frac{1}{2} + \frac{1}{2} (-1)^{N-1} = \begin{cases} 1 & \text{if } N \text{ odd} \\ 0 & \text{if } N \text{ even} \end{cases}$$

Note that $\lim_{N \rightarrow \infty} S_N$ does not exist, so that the given series diverges. (We also could use Theorem 2).

10.2.15

Use Theorem 2 to prove that the following series diverges.

$$\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$$

If we let $a_n = \sqrt{n+1} - \sqrt{n}$, then $\lim_{n \rightarrow \infty} a_n = 0$ so we actually can't apply Thm. 2. Instead let's look at partial sums:

$$S_1 = (\sqrt{2} - \sqrt{1}) \quad S_2 = (\sqrt{3} - \sqrt{2}) + (\sqrt{2} - \sqrt{1}) = \sqrt{3} - \sqrt{1}$$

$$\therefore S_N = \sum_{n=1}^N (\sqrt{n+1} - \sqrt{n}) = \sqrt{N+1} - \sqrt{1} = \sqrt{N+1} - 1$$

Note that $\lim_{N \rightarrow \infty} S_N$ diverges; therefore, so does the given series.

In Exercises 21, 25, and 27, use the formula for the sum of a geometric series to find the sum or state that the series diverges.

10.2.21

$$\begin{aligned}\sum_{n=3}^{\infty} \frac{3^n}{11^n} &= \sum_{n=3}^{\infty} \left(\frac{3}{11}\right)^n \\ &= \sum_{n=3}^{\infty} \left(\frac{3}{11}\right)^3 \left(\frac{3}{11}\right)^{n-3} \\ &= \left(\frac{3}{11}\right)^3 \sum_{n=3}^{\infty} \left(\frac{3}{11}\right)^{n-3} \\ &= \left(\frac{3}{11}\right)^3 \sum_{n=0}^{\infty} \left(\frac{3}{11}\right)^n \\ &= \left(\frac{3}{11}\right)^3 \cdot \frac{1}{1 - 3/11} \\ &= \left(\frac{3}{11}\right)^3 \cdot \frac{1}{8/11} \\ &= \left(\frac{3}{11}\right)^3 \cdot \frac{11}{8} \\ &= \frac{3^3}{8 \cdot 11^2} \\ &= \boxed{\frac{27}{968}}\end{aligned}$$

10.2.25

$$\begin{aligned}\sum_{n=2}^{\infty} e^{3-2n} &= \sum_{n=2}^{\infty} e^3 (e^{-2n}) \\ &= e^3 \sum_{n=2}^{\infty} e^{-2n} \\ &= e^3 \sum_{n=2}^{\infty} (e^{-2})^n \\ &= e^3 \sum_{n=2}^{\infty} (e^{-2})^2 (e^{-2})^{n-2} \\ &= e^3 \sum_{n=2}^{\infty} (e^{-4}) (e^{-2})^{n-2} \\ &= e^3 \cdot e^{-4} \sum_{n=0}^{\infty} (e^{-2})^{n-2} \\ &= \frac{1}{e} \cdot \frac{1}{1-e^{-2}} \\ &= \frac{1}{e} \cdot \frac{1}{1-1/e^2} \\ &= \frac{1}{e} \cdot \frac{1}{(e^2-1)/e^2} \\ &= \frac{1}{e} \cdot \frac{e^2}{e^2-1} \\ &= \boxed{\frac{e}{e^2-1}}\end{aligned}$$

10.2.27

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{93^n + 4^{n-2}}{5^n} &= \sum_{n=0}^{\infty} \frac{93^n}{5^n} + \sum_{n=0}^{\infty} \frac{4^{n-2}}{5^n} \\ &= \sum_{n=0}^{\infty} \left(\frac{93}{5}\right)^n + \sum_{n=0}^{\infty} 4^{-2} \cdot \left(\frac{4}{5}\right)^n \\ &= \sum_{n=0}^{\infty} \left(\frac{93}{5}\right)^n + \frac{4^{-2}}{1 - 4/5} \end{aligned}$$

This holds if $\sum (93/5)^n$ converges, but $|93/5| > 1$ so it diverges and hence the original series diverges as well.

Note: The method used in the previous three problems of factoring and rearranging terms to fit the formula for a geometric series is a bit finicky but usually works in general.

10.2.32

Which of the following are geometric series?

$$(a) \sum_{n=0}^{\infty} \frac{7^n}{29^n} = \sum_{n=0}^{\infty} \left(\frac{7}{29}\right)^n \rightarrow \text{GEOMETRIC}$$

$$(b) \sum_{n=3}^{\infty} \frac{1}{n^4} \rightarrow \text{NOT GEOMETRIC}$$

$$(c) \sum_{n=6}^{\infty} \frac{n^2}{2^n} \rightarrow \text{NOT GEOMETRIC}$$

$$\begin{aligned} (d) \sum_{n=5}^{\infty} \pi^{-n} &= \sum_{n=5}^{\infty} \left(\frac{1}{\pi}\right)^n = \sum_{n=5}^{\infty} \left(\frac{1}{\pi}\right)^5 \left(\frac{1}{\pi}\right)^{n-5} \\ &= \left(\frac{1}{\pi}\right)^5 \sum_{n=0}^{\infty} \left(\frac{1}{\pi}\right)^n \rightarrow \text{GEOMETRIC} \end{aligned}$$

10.2.36

Compute the total area of the (infinitely many) triangles in Figure 3.

We need a formula for A_n , the area of the n^{th} triangle (working right to left). We then will have

$$\text{Total Area} = \sum_{n=0}^{\infty} A_n$$

We have that $A_n = \frac{1}{2} \cdot \text{base} \cdot \text{height}$. For each n the height is $\frac{1}{2}$; however the base is given by

$$b(n) = \frac{1}{2^{n+1}}$$

thus we have $A_n = (\frac{1}{2}) \cdot (\frac{1}{2^{n+1}}) \cdot (\frac{1}{2}) = \frac{1}{2^{n+3}}$ and

$$\begin{aligned} \text{Total Area} &= \sum_{n=0}^{\infty} \frac{1}{2^{n+3}} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+3} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^n \\ &= \frac{1}{8} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \\ &= \frac{1}{8} \cdot \frac{1}{1-\frac{1}{2}} \\ &= \frac{1}{8} \cdot \frac{1}{\frac{1}{2}} \\ &= \boxed{\frac{1}{4}} \end{aligned}$$

